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# USING SOME FUZZY AGGREGATION OPERATORS FOR MULTI-OBJECTIVE LINEAR TRANSPORTATION PROBLEM

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**ABSTRACT**

Using fuzzy aggregation operators, compensatory fuzzy approaches can be proposed for multi-objective problems. The variety of operators for the aggregation of objectives might be confusing and might make it difficult to decide which one to apply to the problem. For example while Zimmermann's "min" operator provides numerical efficiency, it does not guarantee compensatory and Pareto-optimality. In this paper, we present brief information about some important compensatory fuzzy aggregation operators and then apply them to the Multi-objective Linear Transportation Problem (MOLTP) to obtain a compensatory compromise Pareto-optimal solution set. And an illustrative example is provided to compare these aggregation operators and to conclude which operator is more appropriate for the concerning problem.
**Keywords:** Transportation problem, multi-objective programming, fuzzy aggregation operators, fuzzy mathematical programming.

**BAZI BULANIK BİRLEŞTİRME OPERATÖRLERİNİN ÇOK AMAÇLI LİNEER TAŞIMA PROBLEMİ İÇİN KULLANILMASI**

**ÖZET**

Bulanık operatörler kullanılarak, çok amaçlı problemler için dengeleyici bulanık yaklaşımlar üretilmektedir. Amaçların birleştirilmesi için kullanılan bu operatörlerin çeşitliliği kafa karıştırıcı olabilir ve probleme hangisinin uygulanacağı kararının verilmesini güçleştirebilir. Örneğin Zimmermann'ın "min" operatörü sayısal hesaplamalarda kolaylık sağlarken, dengeleyici olma özelliğini ve Pareto-optimalliği garanti etmemektedir. Çalışmamızda, bazı önemli bulanık birleştirme operatörleri hakkında temel bilgi sunularak, bu operatörler dengeleyici uzlaşık Pareto-optimal çözüm kümesini elde etmek amacıyla çok amaçlı lineer taşıma problemine uygulanmıştır. Ayrıca bu birleştirme operatörlerini karşılaştırmak amacıyla sayısal bir örnek verilmiş ve ele alınan problem için hangi operatörün uygun olduğu hakkında sonuçlar vurgulanmıştır.
**Anahtar Sözcükler:** Taşıma problemi, çok amaçlı programlama, bulanık birleştirme operatörleri, bulanık matematiksel programlama.

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## 1. INTRODUCTION

Transportation Problem (TP) has wide practical applications in logistic systems, manpower planning, personnel allocation, inventory control, production planning, etc. and aims to find the

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best way to fulfill the demand of $n$ demand points using the capacities of $m$ supply points. In many real-life situations, decisions are often made in the presence of multiple, conflicting, incommensurate objectives. Thus, MOLTP becomes more useful and includes objectives such as distribution cost, quantity of goods delivered, unfulfilled demand, average delivery time of the commodities, reliability of transportation, accessibility to the users, product deterioration, etc.

After Lee and Moore [1] studied the optimization of transportation problems with multiple objectives, Diaz [2, 3] and Isermann [4] proposed procedures to generate all non-dominated solutions to the MOLTP. Current et al. [5, 6] did a review of multi-objective design of transportation networks. Climaco et al. [7] and Ringuest et al. [8] developed interactive algorithms for the MOLTP. Bit et al. [9] presented an additive fuzzy programming model for the MOLTP. Some solution procedures for MOLTP where the cost coefficients of the objective functions, and the source and destination parameters expressed as interval values by the decision maker are proposed by Das et al. [10] and Ahlatcioglu et al. [11]. Li and Lai [12] and Wahed [13] proposed a fuzzy compromise programming approach to MOTP. Basing on extension principle, Liu and Kao [14] developed a procedure to derive the fuzzy objective value of the fuzzy transportation problem where the cost coefficients, supply and demand quantities are fuzzy numbers. Using signed distance ranking, defuzzification by signed distance, interval-valued fuzzy sets and statistical data, Chiang [15] get the transportation problem in the fuzzy sense. Ammar and Youness [16] examined the solution of multi objective TP which has fuzzy cost, source and destination parameters. They introduced the concepts of fuzzy efficient and parametric efficient solutions. And Barough [17] presented a two stage procedure for fuzzy transportation problem in which the cost coefficients and supply and demand quantities are fuzzy numbers. Ojha et al. [18] formulated single and multi-objective transportation models with fuzzy relations under the fuzzy logic. In that paper, the parameters of models are stated by verbal words such as 'very high', 'high', 'medium', 'low' and 'very low'. And both models are solved with Real coded Genetic Algorithms. Gupta and Kumar [19] is proposed a new method to find solution of a MOLTP by representing all the parameters as interval-valued fuzzy numbers. Ojha et al. [20] introduced the modified subgradient method for optimization and its effectiveness in a fuzzy transportation model. Here a multi-item balanced transportation problem is formulated where unit transportation costs, available spaces and budgets at destinations are imprecise.

In the most of these notable studies from the literature, Zimmermann's "min" operator is used to aggregate the multiple objectives. And as far as we know, the efficiency of the aggregation operators for the solution of MOLTP has not been studied yet. So, in this paper, using some important fuzzy aggregation operators, we present some compensatory fuzzy approaches to MOLTP. By means of a numerical example, we also conclude which operator is more appropriate for the concerning problem.

This paper is organized as follows. Next section provides brief information about compensatory fuzzy aggregation operators. Section 3 explains our methodology using Werners' compensatory "fuzzy and'' and "fuzzy or'' operator, Modified Zimmermann's convex combination of the min- and max-operators, Lai and Hwang's augmented max–min operator. Section 4 gives an illustrative numerical example. Finally, Section 5 and Section 6 include the comparison results and conclusion.

## 2. COMPENSATORY FUZZY AGGREGATION OPERATORS

The variety of operators for the aggregation of fuzzy sets might be confusing and might make it difficult to decide which one to use in a specific model or situation. Zimmermann [21] proposed the following eight rules to justify a suitable operator for a particular fuzzy decision problem. Criteria for selecting appropriate aggregation operators are axiomatic strength, empirical fit, adaptability, numerical efficiency, compensation, range of compensation, aggregating behavior, required scale level of membership functions.

The most important aspect in the fuzzy approach is the compensatory or non-compensatory nature of the aggregate operator. By compensation [21], in the context of aggregation operators for two fuzzy sets, it means that the following: given that the degree of membership to the aggregated fuzzy set is $\mu_{agg}\left(x_k\right)=z\left(\mu_{\tilde{A}}\left(x_k\right),\mu_{\tilde{B}}\left(x_k\right)\right)=k$. $z$ is compensatory if $\mu_{agg}\left(x_k\right)=k$ is obtainable for a different $\mu_{\tilde{A}}\left(x_k\right)$ by a change in $\mu_{\tilde{B}}\left(x_k\right)$. Several investigators [21, 22, 23, 24] have discussed this aspect.

Using the linear membership function, Zimmermann proposed the "min'' operator model to the multi-objective linear programming problems [25]. It is usually used due to its easy computation. Although the "min'' operator method has been proven to have several nice properties [21], the solution generated by min operator does not guarantee compensatory and Pareto-optimal [26, 27, 28]. The biggest disadvantage of the aggregation operator "min'' is that it is non-compensatory. In other words, the results obtained by the "min'' operator represent the worst situation and cannot be compensated by other members which may be very good. On the other hand, the decision modeled with maximum operator is called fully compensatory in the sense that it achieves the full satisfaction of a single goal.

As a result of experiment made by Zimmermann and Zysno [29], most of the decisions taken in the real world are neither non-compensatory (min operator) nor fully compensatory. So, these operators do not seem to be very suitable for modeling the real world problems in many situations. To overcome this difficulty Zimmermann and Zysno [29] have suggested a class of hybrid operators called compensatory operator with the help of a suitable parameter of compensation $\gamma$. They showed that the "$\gamma-$operator (or "compensatory and'' operator)'' is more adequate in human decision making than operators "min'', "product'', "max'', "weighted geometric mean''. But it is a nonlinear operator and increases the computational difficulties tremendously.

A computationally efficient compensatory operator is Luhandjula's compensatory min-bounded sum operator: $\mu_D=\gamma\min_i \mu_i+\left(1-\gamma\right)\min\left[1,\sum_i \mu_i\right]$ is presented to solve Multi-objective Linear Programming (MOLP) problem [21]. Unfortunately, it is difficult to determine the compensatory coefficient $\gamma$. The solution generated by min-bounded sum operator is not necessarily efficient. However, it is an attractive one from the standpoint of computational efficiency. In order to overcome this drawback, Li [22] proposed a two-phase approach to overcome this difficulty. As a matter of fact, the first phase is to use Zimmermann's approach. If the possible solution is unique in phase one, it will be a Pareto-optimal solution. Otherwise, in phase two, a new program will be formulated to maximize the arithmetic mean value of all memberships restricted by original constraints and constraints comes from phase one. Obviously, phase two yields an efficient solution because of full compensation of the "averaging'' operator. Chen and Chou [30] proposed a fuzzy approach to integrate the min operator, average operator and two-phase methods. Guu and Wu [31] proposed a similar two-phase model for fuzzy linear programming problem to improve the dominated solution yielded by min operator. To the case of MOLP, Lee and Li [27] associated a two-phase approach with $\alpha-$cut to treat the possibilistic distributions of fuzzy coefficients. Wu and Guu [28] proposed a simplified two-phase model for MOLP to yield a fuzzy efficient solution between non-compensatory ("min'' operator) and full compensatory (average operator). Tiryaki [32] proposed interactive compensatory fuzzy programming for decentralized multi-level linear programming problems to obtain a preferred compensatory compromise Pareto-optimal solution.

In this paper, we will use Werners' compensatory "fuzzy and'' operator and show that the solutions generated by this operator do guarantee Pareto-optimality for our MOLTP problem. And also we will compare this operator with the following other computationally efficient compensatory fuzzy aggregation operators.

Let us introduce these operators, where $0 \le \mu_i \le 1$, $i = 1, 2, \ldots, m$ and the magnitude of $\gamma \in [0,1]$ represent the grade of compensation.

***Werners' compensatory "fuzzy and'' and "fuzzy or'' operators*:** Based on the $\gamma-$operator, Werners [33] introduced the compensatory "fuzzy and'' and "fuzzy or'' operators which are the convex combinations of min and arithmetical mean, and max and arithmetical mean, respectively:

$$\mu_{and} = \gamma \min_i (\mu_i) + \frac{(1-\gamma)}{m} \left( \sum_i \mu_i \right), \tag{1}$$

$$\mu_{or} = \gamma \max_i (\mu_i) + \frac{(1-\gamma)}{m} \left( \sum_i \mu_i \right). \tag{2}$$

Although these operators are not inductive and associative, they are commutative, idempotent, strictly monotonic increasing (and decreasing, respectively) in each component, continuous and compensatory. Obviously, when $\gamma = 1$, these equations reduce to $\mu_{and} = \min$ and $\mu_{or} = \max$, respectively. The combination of these two operators forms the generalized "and'' and "or'' operators.

***Modified Zimmermann's convex combination of the min- and max-operators*:** This compensatory operator is modified by Lai and Hwang and a modified version of Zimmermann and Zysno's $\gamma-$operator [22]:

$$\mu_D = \gamma \min_i \mu_i + (1-\gamma) \max_i \mu_i . \tag{3}$$

***Lai and Hwang's augmented max–min operator* [22]:**

$$\mu_D = \min_i \mu_i + \delta \sum_i \mu_i , \tag{4}$$

where $\delta$ is a sufficiently small positive number. As seen, the augmented max–min operator is an extension of Zimmermann's "min'' operator.

## 3. USING SOME FUZZY AGGREGATION OPERATORS FOR MOLTP

The mathematical model of the MOLTP can be written as follows:

$$\min \quad F^k(\mathbf{x}) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}^k x_{ij} , \; k = 1, 2, \ldots, K , \tag{5}$$

s.t.: $\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1,2,\ldots,m$,

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1,2,\ldots,n,$$

$\forall\, x_{ij} \geq 0,\ i = 1,2,\ldots,m;\ j = 1,2,\ldots,n$.

$x_{ij}$ is decision variable which refers to product quantity that transported from supply point *i* to demand point *j*. $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_n$ are $m$ supply and $n$ demand quantities, respectively. $K$ is the number of the objective functions of MOLTP. $c_{ij}^{k}$ is unit transportation cost from supply point *i* to demand point *j* for the objective $k, (k = 1,2,\ldots,K)$.

Without loss of generality, we assume that $a_i > 0(\forall i),\ b_j > 0(\forall j),\ c_{ij}^{k} \geq 0\ (\forall (i,j))$ and $\sum_i a_i = \sum_j b_j$ (Balance condition).

Now, in the context of multi-objective, let us give the definitions of efficient or non-dominated or Pareto-optimal solutions for MOLTP. These are used instead of the optimal solution concept in a single objective transportation problem.

**Definition 3.1.** (Pareto-optimal Solution for MOLTP). Let *S* be the feasible region of (5). $x^* \in S$ is said to be a Pareto-optimal (strongly-efficient) solution if and only if there does not exist another $x \in S$ such that $F^k(x) \leq F^k(x^*)$ for all *k* and $F^k(x) \neq F^k(x^*)$ for at least one *k,* where $x^* = \{x_{ij}\}$.

**Definition 3.2.** (Compromise solution for MOLTP) A feasible solution $\mathbf{x}^* \in S$ is called a compromise solution of (5) if and only if $\mathbf{x}^* \in E$ and $F^k(\mathbf{x}^*) \leq \underset{\mathbf{x} \in S}{\wedge} F(\mathbf{x})$ where $F(\mathbf{x}) = \left(F^1(\mathbf{x}), F^2(\mathbf{x}), \ldots, F^k(\mathbf{x})\right)$, $\wedge$ stands for "min" operator and $E$ is the set of Pareto-optimal solutions of MOLTP.

### 3.1. Constructing the Membership Functions of Objectives

The membership functions of the objectives will be defined to apply our approach. Let $L_k$ and $U_k$ be the lower and upper bounds of the objective function $F^k$, respectively. In the literature, there are two common ways of determining these bounds ([21]). The first way: Solve the MOLTP as a single objective TP using each time only one objective and ignoring all others. Determine the corresponding values for every objective at each solution derived. And find the best $(L_k)$ and the worst $(U_k)$ values corresponding to the set of solutions. And the second way: By solving *2K* single-objective TP, the lower and upper bounds $L_k$ and $U_k$ can also be determined for each objective $F^k\left(\mathbf{x}\right)$, $k=1,2,\ldots,K$ as follows:

$$L_k=\min_{\mathbf{x}\in S} F^k\left(\mathbf{x}\right), \qquad U_k=\max_{\mathbf{x}\in S} F^k\left(\mathbf{x}\right), \tag{6}$$

Here, we note that (6) will be used for determining the lower and upper bounds of objectives. Also for the sake of simplicity, in this paper we used the linear membership function:

$$\mu_k\left(F^k\right)=\begin{cases} 1, & F^k<L_k, \\ \dfrac{U_k-F^k}{U_k-L_k}, & L_k\le F^k\le U_k, \\ 0, & F^k>U_k. \end{cases} \tag{7}$$

Here, $L_k\neq U_k$, $k=1,2,\ldots,K$ and in the case of $L_k=U_k$, $\mu_k(F^k(\mathbf{x}))=1$. The membership function $\mu_k\left(F^k\right)$ is linear and strictly monotone decreasing for $F^k$ in the interval $\left[L_k,U_k\right]$.

Using Zimmermann's minimum operator ([25]), MOLTP can be written as:

$$\max_x \min_k \mu_k(F^k(\mathbf{x})) \tag{8}$$

s.t. $\mathbf{x}\in S$.

By introducing an auxiliary variable $\lambda$, (8) can be transformed into the following equivalent conventional linear programming problem:

$$\begin{aligned} &\max \quad \lambda \\ &\text{s.t.} \quad \mu_k(F^k(\mathbf{x}))\ge\lambda, \quad k=1,\ldots,K \\ &\mathbf{x}\in S, \ \lambda\in\left[0,1\right]. \end{aligned} \tag{9}$$

Here, we note that (9) is the "min" operator model for MOLTP, and also a nonlinear programming model. Its optimal objective value denotes the maximizing value of the least satisfaction level among all objectives of MOLTP. And it can also be interpreted as the "most basic satisfaction" that each objective in the transportation system can attain.

Now, we can construct the compensatory models with fuzzy aggregation operators for MOLTP as follows:

### 3.2. Werners' Compensatory "fuzzy and" Operator for MOLTP

It is pointed out that Zimmermann's min operator model doesn't always yield a Pareto-optimal solution [26, 27, 28]. By using Werners' $\mu_{and}$ operator ((1)), (9) is converted to:

$$\max \mu_{and}=\lambda+\frac{(1-\gamma)}{K}(\lambda_1+\lambda_2+\ldots+\lambda_K) \tag{10}$$

s.t. $\mathbf{x}\in S$,

$\mu_k\left(F^k\left(\mathbf{x}\right)\right) \geq \lambda + \lambda_k$ ,

$\lambda + \lambda_k \leq 1$ ,

$\lambda, \forall \lambda_k \in [0,1], \;\; k = 1, 2, ..., K$

$\gamma \in [0,1]$ .

So, our compensatory model generates compensatory compromise Pareto-optimal solutions for MOLTP.

We shall give this assertion in the following theorem.

**Theorem:** If $\left(\mathbf{x}, \boldsymbol{\lambda}^{\mathbf{x}}\right)$ is an optimal solution of problem (10), then $\mathbf{x}$ is a Pareto-optimal solution for MOLTP, where $\boldsymbol{\lambda}^{\mathbf{x}} = (\lambda^{\mathbf{x}}, \lambda_1^{\mathbf{x}}, \lambda_2^{\mathbf{x}}, ..., \lambda_K^{\mathbf{x}})$.

**Proof:** The theorem can be proven similarly to ones in [32].

If required, Pareto-optimality test ([34]) can also be applied to the solutions of (10) and it could be seen that these solutions are Pareto-optimal for MOLTP.

### 3.3. Werners' Compensatory "fuzzy or" Operator for MOLTP

By using Werners' $\mu_{or}$ operator ((2)), (9) is converted to as follows:

$$\max \mu_{or} = \alpha - \frac{(1-\gamma)}{K}(\alpha_1 + \alpha_2 + ... + \alpha_K)$$

s.t. $\mathbf{x} \in S$ ,

$\mu_k\left(F^k\left(\mathbf{x}\right)\right) = \alpha - \alpha_k$ , $\forall\, k = 1, 2, ..., K$

$\mu_k\left(F^k\left(\mathbf{x}\right)\right) \geq \alpha$ , for at least one $k \in \{1, 2, \ldots, K\}$

$0 \leq \alpha_k \leq \alpha \leq 1, \;\; \forall k = 1, 2, ..., K$

$\gamma \in [0,1]$

or

$$\max \mu_{or} = \alpha - \frac{(1-\gamma)}{K}(\alpha_1 + \alpha_2 + ... + \alpha_K), \tag{11}$$

s.t. $\mathbf{x} \in S$ ,

$\mu_k\left(F^k\left(\mathbf{x}\right)\right) \geq \alpha - \alpha_k$ , $\forall\, k = 1, 2, ..., K$ ,

$\alpha - \alpha_k \leq 1$ , $\forall\, k = 1, 2, ..., K$ ,

$\mu_k\left(F^k\left(\mathbf{x}\right)\right) + M\, r_k \geq \alpha$ , $\forall\, k = 1, 2, ..., K$ ,

$\sum_{k=1}^{K} r_k \leq K - 1$ ,

$0 \leq \alpha_k \leq \alpha \leq 1, \;\; \forall k = 1, 2, ..., K$ ,

$r_k \in \{0, 1\}$ , $\forall k = 1, 2, \ldots, K$ ,

$\gamma \in [0,1]$.

Objective function of (11) maximizes the linear combination of the level of satisfaction of the most satisfied objective (max operator) and the level of satisfaction average satisfied objective.

### 3.4. *Modified Zimmermann's Convex Combination of the min- and max-operators* for MOLTP

With modified Zimmermann's convex combination of the min- and max-operators

$$\mu_D = \gamma \min_k \mu_k\left(F^k(\mathbf{x})\right) + (1-\gamma)\max_k \mu_k\left(F^k(\mathbf{x})\right),$$

our MOLTP becomes

$$\max_{\mathbf{x}\in S}\left\{\gamma \min_k \mu_k\left(F^k(\mathbf{x})\right) + (1-\gamma)\max_k \mu_k\left(F^k(\mathbf{x})\right)\right\}$$

or

$$\max\left\{\gamma\alpha_1 + (1-\gamma)\alpha_2\right\}$$

s.t. $\mu_k\left(F^k(\mathbf{x})\right) \ge \alpha_1$, $\forall k = 1,2,\dots,K$

$\mu_k\left(F^k(\mathbf{x})\right) \ge \alpha_2$, for at least one $k \in \{1,2,\dots,K\}$

$\mathbf{x}\in S$, $\alpha_1,\alpha_2 \in [0,1]$

or

$$\max\left\{\gamma\alpha_1 + (1-\gamma)\alpha_2\right\} \tag{12}$$

s.t. $\mu_k\left(F^k(\mathbf{x})\right) \ge \alpha_1$, $\forall k = 1,2,\dots,K$,

$\mu_k\left(F^k(\mathbf{x})\right) + Mr_k \ge \alpha_2$, $\forall k = 1,2,\dots,K$,

$\sum_{k=1}^{K} r_k \le K-1$,

$\mathbf{x}\in S$, $\alpha_1,\alpha_2 \in [0,1]$,

$r_k \in \{0,1\}$, $\forall k = 1,2,\dots,K$

where $M$ is a very large real number.

The objective function of (12) maximizes the linear combination of the level of satisfaction of the less satisfied objective (min operator) and the level of satisfaction of the most satisfied objective (max operator).

### 3.5. Lai and Hwang's Augmented max–min Operator for MOLTP

Using (4), the modified Lai and Hwang's augmented max-min operator for MOLTP must be

$$\mu_D = \min_k \mu_k\left(F^k(\mathbf{x})\right) + \delta\sum_k \mu_k\left(F^k(\mathbf{x})\right).$$

Taking $\min_k \mu_k\left(F^k(\mathbf{x})\right)=\lambda$, our MOLTP becomes

$$\max \mu_D = \max\left\{\lambda+\delta\sum_k \mu_k\left(F^k(\mathbf{x})\right)\right\}$$

$$\text{s.t.} \quad \mu_k\left(F^k(\mathbf{x})\right)\geq\lambda,\ \forall k=1,2,\ldots,K, \tag{13}$$

$$\mathbf{x}\in S.$$

### 3.6. A Hybrid Approach of Werners' and Lai-Hwang's Operators for MOLTP

If we use $\min_k \mu_k\left(F^k(\mathbf{x})\right)=\lambda \Rightarrow \mu_k\left(F^k(\mathbf{x})\right)=\lambda+\lambda_k$ by Werners' sense and combine it with Lai-Hwang's augmented max-min operator, our MOLTP will become

$$\max\left\{(1+\delta)\lambda+\delta\sum_{k=1}^{K}\lambda_k\right\} \tag{14}$$

$$\text{s.t.} \quad \mu_k\left(F^k(\mathbf{x})\right)\geq\lambda+\lambda_k,\ \forall k=1,2,\ldots,K,$$

$$\lambda+\lambda_k\leq 1,\ \forall k=1,2,\ldots,K$$

$$\mathbf{x}\in S.$$

## 4. AN ILLUSTRATIVE EXAMPLE

Let us consider a multiple-objective transportation problem with the following characteristics:

Supplies: $a_1=5$, $a_2=4$, $a_3=2$, $a_4=9$.

Demands: $b_1=4$, $b_2=4$, $b_3=6$, $b_4=2$, $b_5=4$.

Penalties: $C^1=\begin{bmatrix}9&12&9&6&9\\7&3&7&7&5\\6&5&9&11&3\\6&8&11&2&2\end{bmatrix}$ $C^2=\begin{bmatrix}2&9&8&1&4\\1&9&9&5&2\\8&1&8&4&5\\2&8&6&9&8\end{bmatrix}$ $C^3=\begin{bmatrix}2&4&6&3&6\\4&8&4&9&2\\5&3&5&3&6\\6&9&6&3&1\end{bmatrix}$

The lower and upper bounds of the objectives that we obtained to define the membership functions of the objectives are shown in Table 1.

**Table 1.** Bound values of objectives

| | $F^1$ | $F^2$ | $F^3$ |
|---|---|---|---|
| $L_k$ | 102 | 72 | 64 |
| $U_k$ | 188 | 157 | 136 |

From Table 1, the membership functions are obtained as follows:

$$\mu_1(x) = \frac{188 - F^1(x)}{188 - 102} = \frac{188 - F^1(x)}{86},$$

$$\mu_2(x) = \frac{157 - F^2(x)}{157 - 72} = \frac{157 - F^2(x)}{85},$$

$$\mu_3(x) = \frac{136 - F^3(x)}{136 - 64} = \frac{136 - F^3(x)}{72}.$$

**4.1. Werners' Compensatory "fuzzy and" Operator for the Example**

Using (10), our compensatory problem will be in the form as follows:

$$\max \mu_{and} = \lambda + \frac{(1-\gamma)}{3}(\lambda_1 + \lambda_2 + \lambda_3) \tag{15}$$

$$\left.\begin{array}{l} \sum_{j=1}^{5} x_{1j} = 5,\ \sum_{j=1}^{5} x_{2j} = 4,\ \sum_{j=1}^{5} x_{3j} = 2, \\ \sum_{j=1}^{5} x_{4j} = 9, \sum_{i=1}^{4} x_{i1} = 4,\ \sum_{i=1}^{4} x_{i2} = 4, \\ \sum_{i=1}^{4} x_{i3} = 6,\ \sum_{i=1}^{4} x_{i4} = 2, \sum_{i=1}^{4} x_{i5} = 4, \\ x_{ij} \geq 0,\ i = 1,2,3,4.\ \ j = 1,2,3,4,5, \end{array}\right\} \mathbf{x} \in S$$

$$\mu_1(x) \geq \lambda + \lambda_1,\ \mu_2(x) \geq \lambda + \lambda_2,\ \mu_3(x) \geq \lambda + \lambda_3,$$

$$\lambda + \lambda_1 \leq 1,\ \lambda + \lambda_2 \leq 1,\ \lambda + \lambda_3 \leq 1,$$

$$\lambda, \lambda_1, \lambda_2, \lambda_3 \geq 0.$$

By solving (15), the results for different 11 values of the compensation parameter $\gamma$ with $0.1$ increment are obtained and given in Table 2(a) and Table 2(b). The results are: the compensation satisfactory level $\mu_{and}$, the values of objective functions $F^k\ (k = 1,2,3)$; the satisfactory levels of the objectives corresponding to solution $\mathbf{x}$, (i.e. the values of membership functions) $\mu_k\ (k = 1,2,3)$; the most basic satisfactory level $\lambda$; respectively.

As it can be seen from Table 2(a) and Table 2(b), our compensatory model generates the following compensatory compromise Pareto-optimal solutions $\mathbf{X}^{1*}$, $\mathbf{X}^{2*}$ and $\mathbf{X}^{3*}$ for this example.

**Table 2(a).** The results of our compensatory model with $\mu_{and}$ .

| | $\gamma = 0$ | $\gamma = 0.1$ | $\gamma = 0.2$ | $\gamma = 0.3$ | $\gamma = 0.4$ | $\gamma = 0.5$ | $\gamma = 0.6$ | $\gamma = 0.7$ | $\gamma = 0.8$ | $\gamma = 0.9$ | $\gamma = 1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mu_{and}$ | 0.7220 | 0.7147 | 0.7099 | 0.7054 | 0.7054 | 0.7054 | 0.7054 | 0.7054 | 0.7054 | 0.7054 | 0.7054 |
| $F^1$ | 127 | 130.206 | 130.206 | 127.332 | 127.332 | 127.332 | 127.332 | 127.332 | 127.332 | 127.332 | 127.332 |
| $F^2$ | 104 | 99.878 | 99.878 | 97.037 | 97.037 | 97.037 | 97.037 | 97.037 | 97.037 | 97.037 | 97.037 |
| $F^3$ | 76 | 77.374 | 77.374 | 85.208 | 85.208 | 85.208 | 85.208 | 85.208 | 85.208 | 85.208 | 85.208 |
| $\mu_1$ | 0.7093 | 0.6720 | 0.6720 | 0.7054 | 0.7054 | 0.7054 | 0.7054 | 0.7054 | 0.7054 | 0.7054 | 0.7054 |
| $\mu_2$ | 0.6235 | 0.6720 | 0.6720 | 0.7054 | 0.7054 | 0.7054 | 0.7054 | 0.7054 | 0.7054 | 0.7054 | 0.7054 |
| $\mu_3$ | 0.8333 | 0.81425 | 0.81425 | 0.7054 | 0.7054 | 0.7054 | 0.7054 | 0.7054 | 0.7054 | 0.7054 | 0.7054 |
| $\lambda$ | 0.0624 | 0.6720 | 0.6720 | 0.7054 | 0.7054 | 0.7054 | 0.7054 | 0.7054 | 0.7054 | 0.7054 | 0.7054 |

**Table 2(b).** The results of our compensatory model with $\mu_{and}$ .

| | $\gamma = 0$ | $\gamma = 0.1$ | $\gamma = 0.2$ | $\gamma = 0.3$ | $\gamma = 0.4$ | $\gamma = 0.5$ | $\gamma = 0.6$ | $\gamma = 0.7$ | $\gamma = 0.8$ | $\gamma = 0.9$ | $\gamma = 1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\chi_{11}$ | 3 | 3 | 3 | 1.5242 | 1.5242 | 1.5242 | 1.5242 | 1.5242 | 1.5242 | 1.5242 | 1.5242 |
| $\chi_{12}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\chi_{13}$ | 0 | 0 | 0 | 1.4758 | 1.4758 | 1.4758 | 1.4758 | 1.4758 | 1.4758 | 1.4758 | 1.4758 |
| $\chi_{14}$ | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| $\chi_{15}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\chi_{21}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\chi_{22}$ | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| $\chi_{23}$ | 2 | 1.542 | 1.542 | 0.8984 | 0.8984 | 0.8984 | 0.8984 | 0.8984 | 0.8984 | 0.8984 | 0.8984 |
| $\chi_{24}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\chi_{25}$ | 0 | 0.4580 | 0.4580 | 1.1016 | 1.1016 | 1.1016 | 1.1016 | 1.1016 | 1.1016 | 1.1016 | 1.1016 |
| $\chi_{31}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\chi_{32}$ | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| $\chi_{33}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\chi_{34}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\chi_{35}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\chi_{41}$ | 1 | 1 | 1 | 2.4758 | 2.4758 | 2.4758 | 2.4758 | 2.4758 | 2.4758 | 2.4758 | 2.4758 |
| $\chi_{42}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\chi_{43}$ | 4 | 4.4580 | 4.4580 | 3.6258 | 3.6258 | 3.6258 | 3.6258 | 3.6258 | 3.6258 | 3.6258 | 3.6258 |
| $\chi_{44}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\chi_{45}$ | 4 | 3.5420 | 3.5420 | 2.8984 | 2.8984 | 2.8984 | 2.8984 | 2.8984 | 2.8984 | 2.8984 | 2.8984 |

For $\gamma = 0 \Rightarrow \mathbf{X}^{1*} = \begin{bmatrix} 3 & 0 & 0 & 2 & 0 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 4 & 0 & 4 \end{bmatrix}$ , $F^1\left(X^{1*}\right) = 127$, $F^2\left(X^{1*}\right) = 104$,
$F^3\left(X^{1*}\right) = 76.$

For $\gamma = 0.1$ and $\gamma = 0.2$ $\Rightarrow \mathbf{X}^{2*} = \begin{bmatrix} 3 & 0 & 0 & 2 & 0 \\ 0 & 2 & 1.542 & 0 & 0.4580 \\ 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 4.4580 & 0 & 3.5420 \end{bmatrix}$ ,

$F^1\left(X^{2*}\right) = 130.2060$, $F^2\left(X^{2*}\right) = 99.8780$, $F^3\left(X^{2*}\right) = 77.3740$.

From $\gamma = 0.3$ to $\gamma = 1.0 \Rightarrow \mathbf{X}^{3*} = \begin{bmatrix} 1.5242 & 0 & 1.4758 & 2 & 0 \\ 0 & 2 & 0.8984 & 0 & 1.1016 \\ 0 & 2 & 0 & 0 & 0 \\ 2.4758 & 0 & 3.6258 & 0 & 2.8984 \end{bmatrix}$ ,

$F^1\left(X^{3*}\right) = 127.3320$, $F^2\left(X^{3*}\right) = 97.0370$, $F^3\left(X^{3*}\right) = 85.2080$.

All of these solutions pointed out that the certainly transported amounts are:

$$\left\{ \begin{matrix} x_{12} = x_{15} = x_{21} = x_{24} = x_{31} = x_{33} = x_{34} = x_{35} = x_{42} = x_{44} = 0, \\ x_{14} = x_{22} = x_{32} = 2. \end{matrix} \right\}.$$

And also, the least transported amount are:

$$\left\{ \begin{matrix} x_{11} \geq 1.5242, x_{23} \geq 0.8984, \\ x_{41} \geq 1, \\ x_{43} \geq 3.6258, x_{45} \geq 2.8984 \end{matrix} \right\}.$$

For $\gamma = 0$, $\mu_{and}$ equals to average operator (full-compensatory) operator that is $\mu_{and} = \min \frac{1}{3} \sum_{k=1}^{3} \mu_k\left(F^k\left(\mathbf{x}\right)\right) = 0.7220$ and gives the solution $\mathbf{X}^{1*}$. The satisfactory level of the transportation system for our MOLTP is averagely $0.7220$.

For $\gamma = 1$, $\mu_{and}$ equals to $\min$ (non-compensatory) operator that is $\mu_{and} = \min_k \mu_k\left(F^k\left(\mathbf{x}\right)\right) = 0.7054$ and gives the solution $\mathbf{X}^{3*}$. This solution remains the same from $\gamma = 0.3$ to $\gamma = 1$. As seen the minimal satisfactory level of all objectives is equal to $0.7054$.

**4.2. Werners' Compensatory "fuzzy or" Operator for the Example**

Using (11), our compensatory problem will be in the form as follows:

$$\max \mu_{or} = \alpha - \frac{(1-\gamma)}{3}(\alpha_1 + \alpha_2 + \alpha_3), \tag{16}$$

s.t. $\mathbf{x} \in S$,

$$\mu_1(x) \geq \alpha - \alpha_1,\ \mu_2(x) \geq \alpha - \alpha_2,\ \mu_3(x) \geq \alpha - \alpha_3,$$

$$\alpha - \alpha_1 \leq 1,\ \alpha - \alpha_2 \leq 1,\ \alpha - \alpha_3 \leq 1,$$

$$\mu_1(x) + M\, r_1 \geq \alpha,\ \ \mu_2(x) + M\, r_2 \geq \alpha,\ \ \mu_3(x) + M\, r_2 \geq \alpha\ ,$$

$$r_1 + r_2 + r_3 \leq 2,$$

$$0 \leq \alpha_k \leq \alpha \leq 1,\ \ k = 1,2,3,$$

$$r_k \in \{0,1\},\ k = 1,2,3,$$

$$\gamma \in [0,1].$$

By solving (16), the results for different 11 values of the compensation parameter $\gamma$ with $0.1$ increment are obtained and given in Table 3(a) and Table 3(b).

**Table 3(a).** The results of our compensatory model with $\mu_{or}$ .

| | $\gamma = 0$ | $\gamma = 0.1$ | $\gamma = 0.2$ | $\gamma = 0.3$ | $\gamma = 0.4$ | $\gamma = 0.5$ | $\gamma = 0.6$ | $\gamma = 0.7$ | $\gamma = 0.8$ | $\gamma = 0.9$ | $\gamma = 1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mu_{or}$ | 0.722 | 0.733 | 0.750 | 0.779 | 0.811 | 0.842 | 0.874 | 0.905 | 0.937 | 0.968 | 1.0 |
| $F^1$ | 127 | 127 | 127 | 157 | 157 | 157 | 157 | 157 | 157 | 157 | 134 |
| $F^2$ | 104 | 104 | 123 | 72 | 72 | 72 | 72 | 72 | 72 | 72 | 122 |
| $F^3$ | 76 | 76 | 66 | 86 | 86 | 86 | 86 | 86 | 86 | 86 | 64 |
| $\mu_1$ | 0.7093 | 0.7093 | 0.972 | 0.3605 | 0.3605 | 0.3605 | 0.3605 | 0.3605 | 0.3605 | 0.3605 | 0.6279 |
| $\mu_2$ | 0.6235 | 0.6235 | 0.263 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0.4118 |
| $\mu_3$ | 0.8333 | 0.8333 | 0.572 | 0.6944 | 0.6944 | 0.6944 | 0.6944 | 0.6944 | 0.6944 | 0.6944 | 1 |
| $\alpha$ | 0.833 | 0.833 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

**Table 3(b).** The results of our compensatory model with $\mu_{or}$ .

| | $\gamma = 0$ | $\gamma = 0.1$ | $\gamma = 0.2$ | $\gamma = 0.3$ | $\gamma = 0.4$ | $\gamma = 0.5$ | $\gamma = 0.6$ | $\gamma = 0.7$ | $\gamma = 0.8$ | $\gamma = 0.9$ | $\gamma = 1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\chi_{11}$ | 3 | 3 | 4 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| $\chi_{12}$ | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 |
| $\chi_{13}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\chi_{14}$ | 2 | 2 | 0 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 0 |
| $\chi_{15}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\chi_{21}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| $\chi_{22}$ | 2 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\chi_{23}$ | 2 | 2 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 |
| $\chi_{24}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\chi_{25}$ | 0 | 0 | 0 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 0 |
| $\chi_{31}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\chi_{32}$ | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| $\chi_{33}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\chi_{34}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\chi_{35}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\chi_{41}$ | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 |
| $\chi_{42}$ | 0 | 0 | 0 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 0 |
| $\chi_{43}$ | 4 | 4 | 3 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 3 |
| $\chi_{44}$ | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 |
| $\chi_{45}$ | 4 | 4 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4 |

**Table 4(a).** The results of our compensatory model with *Modified Zimmermann's convex combination of the min- and max-operators*.

| | $\gamma = 0$ | $\gamma = 0.1$ | $\gamma = 0.2$ | $\gamma = 0.3$ | $\gamma = 0.4$ | $\gamma = 0.5$ | $\gamma = 0.6$ | $\gamma = 0.7$ | $\gamma = 0.8$ | $\gamma = 0.9$ | $\gamma = 1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mu_D$ | 1.000 | 0.919 | 0.882 | 0.827 | 0.788 | 0.728 | 0.730 | 0.705 | 0.705 | 0.705 | 0.705 |
| $F^1$ | 157 | 103 | 134 | 140.148 | 140.148 | 117.395 | 131.077 | 127.332 | 127.332 | 127.332 | 127.332 |
| $F^2$ | 72 | 132 | 122 | 109.704 | 109.704 | 103.063 | 100.739 | 97.037 | 97.037 | 97.037 | 97.037 |
| $F^3$ | 86 | 100 | 64 | 68.099 | 68.099 | 90.312 | 76 | 85.208 | 85.208 | 85.208 | 85.208 |
| $\mu_1$ | 0.3605 | 0.9884 | 0.6279 | 0.5564 | 0.5564 | 0.8210 | 0.6619 | 0.7054 | 0.7054 | 0.7054 | 0.7054 |
| $\mu_2$ | 1.000 | 0.2941 | 0.4118 | 0.5564 | 0.5564 | 0.6346 | 0.6619 | 0.7054 | 0.7054 | 0.7054 | 0.7054 |
| $\mu_3$ | 0.6944 | 0.5 | 1.000 | 0.9431 | 0.9431 | 0.6346 | 0.8333 | 0.7054 | 0.7054 | 0.7054 | 0.7054 |
| $\alpha_1$ | 0 | 0.294 | 0.412 | 0.556 | 0.556 | 0.635 | 0.662 | 0.705 | 0.705 | 0.705 | 0.705 |
| $\alpha_2$ | 1.000 | 0.988 | 1.000 | 0.943 | 0.943 | 0.821 | 0.833 | 0.705 | 0.705 | 0.705 | 0 |

**Table 4(b).** The results of our compensatory model with *Modified Zimmermann's convex combination of the min- and max-operators*.

| | $\gamma=0$ | $\gamma=0.1$ | $\gamma=0.2$ | $\gamma=0.3$ | $\gamma=0.4$ | $\gamma=0.5$ | $\gamma=0.6$ | $\gamma=0.7$ | $\gamma=0.8$ | $\gamma=0.9$ | $\gamma=1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\chi_{11}$ | 3 | 0 | 3 | 1.975 | 1.975 | 0 | 3 | 1.524 | 1.524 | 1.524 | 1.524 |
| $\chi_{12}$ | 0 | 0 | 2 | 2 | 2 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\chi_{13}$ | 0 | 5 | 0 | 0 | 0 | 3 | 0 | 1.476 | 1.476 | 1.476 | 1.476 |
| $\chi_{14}$ | 2 | 0 | 0 | 1.025 | 1.025 | 2 | 2 | 2 | 2 | 2 | 2 |
| $\chi_{15}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\chi_{21}$ | 0 | 0 | 1 | 2.025 | 2.025 | 0 | 0.815 | 0 | 0 | 0 | 0 |
| $\chi_{22}$ | 0 | 3 | 0 | 0 | 0 | 2 | 2 | 2 | 2 | 2 | 2 |
| $\chi_{23}$ | 0 | 1 | 3 | 1.975 | 1.975 | 1.229 | 1.185 | 0.898 | 0.898 | 0.898 | 0.898 |
| $\chi_{24}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\chi_{25}$ | 4 | 0 | 0 | 0 | 0 | 0.771 | 0 | 1.102 | 1.102 | 1.102 | 1.102 |
| $\chi_{31}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\chi_{32}$ | 2 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| $\chi_{33}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\chi_{34}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\chi_{35}$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\chi_{41}$ | 1 | 4 | 0 | 0 | 0 | 4 | 0.185 | 2.476 | 2.476 | 2.476 | 2.476 |
| $\chi_{42}$ | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\chi_{43}$ | 6 | 0 | 3 | 4.025 | 4.025 | 1.771 | 4.815 | 3.626 | 3.626 | 3.626 | 3.626 |
| $\chi_{44}$ | 0 | 2 | 2 | 0.975 | 0.975 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\chi_{45}$ | 0 | 3 | 4 | 4 | 4 | 3.229 | 4 | 2.898 | 2.898 | 2.898 | 2.898 |

**4.3.** ***Modified Zimmermann's Convex Combination of the min- and max-operators*** **for the Example**

Using (12), our compensatory problem will be in the form as follows:

$$\max\left\{\gamma\alpha_1+\left(1-\gamma\right)\alpha_2\right\} \tag{17}$$

s.t. $\mathbf{x}\in S$ ,

$\mu_1(x) \geq \alpha_1,\ \mu_2(x) \geq \alpha_1,\ \mu_3(x) \geq \alpha_1,$

$\mu_1(x) + M\, r_1 \geq \alpha_2,\ \ \mu_2(x) + M\, r_2 \geq \alpha_2,\ \ \mu_3(x) + M\, r_2 \geq \alpha_2,$

$r_1 + r_2 + r_3 \leq 2,$

$\alpha_1, \alpha_2 \in [0,1],$

$r_k \in \{0,1\},\ k = 1,2,3$

where $M$ is a very large real number.
The results of (17) are given in Table 4(a) and Table 4(b).

### 4.4. *Lai and Hwang's Augmented max–min Operator for* the Example

Using (13), our compensatory problem will be in the form as follows:

$$\max \mu_D = \max\left\{\lambda + \delta\left(\mu_1(\mathbf{x}) + \mu_2(\mathbf{x}) + \mu_3(\mathbf{x})\right)\right\} \tag{18}$$

s.t. $\mu_1(\mathbf{x}) \geq \lambda$ , $\mu_2(\mathbf{x}) \geq \lambda$ , $\mu_2(\mathbf{x}) \geq \lambda$ ,

$\mathbf{x} \in S$ .

where $\delta = 10^{-5}$ . The results of (18) are given in Table 5.

**Table 5.** The results of our compensatory model with *Lai and Hwang's augmented max–min operator for the example*

| | $\delta = 10^{-5}$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mu_D$ | 0.705 | | $x_{11}$ | 1.524 | | $x_{24}$ | 0 | | $x_{42}$ | 0 |
| $F^1$ | 127.332 | | $x_{12}$ | 0 | | $x_{25}$ | 1.102 | | $x_{43}$ | 3.626 |
| $F^2$ | 97.037 | | $x_{13}$ | 1.476 | | $x_{31}$ | 0 | | $x_{44}$ | 0 |
| $F^3$ | 85.208 | | $x_{14}$ | 2 | | $x_{32}$ | 2 | | $x_{45}$ | 2.898 |
| $\mu_1$ | 0.7054 | | $x_{15}$ | 0 | | $x_{33}$ | 0 | | | |
| $\mu_2$ | 0.7054 | | $x_{21}$ | 0 | | $x_{34}$ | 0 | | | |
| $\mu_3$ | 0.7054 | | $x_{22}$ | 2 | | $x_{35}$ | 0 | | | |
| $\lambda$ | 0.705 | | $x_{23}$ | 0.898 | | $x_{41}$ | 2.476 | | | |

### 4.5. A Hybrid Approach of Werners' and Lai-Hwang's Operators for the Example

Using (14), our compensatory problem will be in the form as follows:

$$\max\left\{(1+\delta)\lambda + \delta\left(\lambda_1 + \lambda_2 + \lambda_3\right)\right\} \tag{19}$$

s.t. $\mu_1(x) \geq \lambda + \lambda_1,\ \mu_2(x) \geq \lambda + \lambda_2,\ \mu_3(x) \geq \lambda + \lambda_3,$

$\lambda + \lambda_1 \leq 1,\ \lambda + \lambda_2 \leq 1,\ \lambda + \lambda_3 \leq 1,$

$\mathbf{x} \in S$

$\lambda, \lambda_1, \lambda_2, \lambda_3 \geq 0.$

where $\delta = 10^{-5}$. The results of (19) are given in Table 6.

Remark: All solutions are obtained by using the GAMS computer package.

**Table 6.** The results of our compensatory model with hybrid approach of Werners' and Lai-Hwang's operators for MOLTP

| | $\delta = 10^{-5}$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mu_D$ | 0.705 | | $x_{11}$ | 1.524 | | $x_{24}$ | 0 | | $x_{42}$ | 0 |
| $F^1$ | 127.332 | | $x_{12}$ | 0 | | $x_{25}$ | 1.102 | | $x_{43}$ | 3.626 |
| $F^2$ | 97.037 | | $x_{13}$ | 1.476 | | $x_{31}$ | 0 | | $x_{44}$ | 0 |
| $F^3$ | 85.208 | | $x_{14}$ | 2 | | $x_{32}$ | 2 | | $x_{45}$ | 2.898 |
| $\mu_1$ | 0.7054 | | $x_{15}$ | 0 | | $x_{33}$ | 0 | | | |
| $\mu_2$ | 0.7054 | | $x_{21}$ | 0 | | $x_{34}$ | 0 | | | |
| $\mu_3$ | 0.7054 | | $x_{22}$ | 2 | | $x_{35}$ | 0 | | | |
| $\lambda$ | 0.705 | | $x_{23}$ | 0.898 | | $x_{41}$ | 2.476 | | | |

## 5. COMPARISON RESULTS

Among several various operators, we selected and used Werners' $\mu_{and}$ operator as a suitable one for MOLTP, basing on Zimmermann's eight rules to justify a suitable operator [21]. These reasons can be given such as: Adaptability: this operator is dependent on the context and the semantic interpretation; that is it models a decision problem which is MOLTP. Thus our proposed fuzzy compensatory method aids the decision maker to get a suitable decision according to the situation; Numerical efficiency: this operator is computationally efficient; Compensation: this operator has compensation if a change in a member of $\mu_{and}$ can be counteracted by a change in an another member of it; Range of resulting membership: the larger the range of resulting membership the better the operator, for example, in Werners' $\mu_{and}$ operator, $\mu_{and} = 0.7054$ for $\gamma = 1$ (it means "min'' operator), $\mu_{and} = 0.7220$ for $\gamma = 0$ (it means "average'' operator). Although modified Zimmermann's approach gives the larger the range of resulting membership, that is, $\mu_D \in [0.7054, 1]$ but it does not guarantee to get Pareto-optimal solution. Although using Werners' $\mu_{or}$ operator, the range of resulting membership is $\mu_D \in [0.7220, 1]$, but while this operator satisfies the full satisfaction of at least one objective, some others' satisfactions may be zero. Whereas Werners' $\mu_{and}$ operator does also guarantee the least degree of satisfactions among all objectives. For this reason, $\mu_{or}$ operator is not appropriate for MOLTP. Lai and Hwang's augmented max–min operator generates a unique Pareto-optimal solution near to the "min'' operator because $\delta$ is sufficiently small positive number, whereas Werners' $\mu_{and}$ operator has more Pareto-optimal solution variety dependent on $\gamma$. And the hybrid of Werners' $\mu_{and}$ and augmented max–min operators also generates a unique Pareto-optimal solution similar to augmented max–min operators' one. Therefore, $\mu_{and}$ operator enables us to choose a compromise solution in a wider set. The compromise solution is both compensatory and Pareto optimal. Using $\mu_{and}$ operator, our method achieves the compromise solution for MOLTP in an only one-phase instead of afore mentioned two phase approaches [24]. And we also gave a theorem that the compensatory solution generated by this operator does guarantee Pareto-optimality for our MOLTP.

## 6. CONCLUSIONS

As known, the solution techniques of MOLTP are often encountered in the literature. It is quite useful using the fuzzy techniques from the point of view efficiency and simplicity. In the literature, it is mostly used Zimmermann's min operator to aggregate multiple objectives. However, it is known that this operator does not guarantee to generate the Pareto-optimal solutions [26, 27, 28]. In this paper, we presented brief information about Werners' compensatory "fuzzy and'' and "fuzzy or'' operators, Modified Zimmermann's convex combination of the min- and max-operators, Lai and Hwang's augmented max–min operator. And we applied them to the MOLTP to obtain a compensatory compromise Pareto-optimal solution set. And an illustrative numerical example is provided to compare these aggregation operators. To investigate the effect of different degrees of compensation, 11 cases with different values of compensations were solved. Among these operators, we conclude that Werners' $\mu_{and}$ operator as a suitable one for MOLTP, basing on Zimmermann's eight rules to justify a suitable operator [21].

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