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γ-Lie structures in γ-prime gamma rings with derivations

Research Article

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Abstract: Let M be a γ-prime weak Nobusawa Γ-ring and $d \neq 0$ be a k-derivation of M such that $k(\gamma) = 0$ and U be a γ-Lie ideal of M. In this paper, we introduce definitions of γ-subring, γ-ideal, γ-prime Γ-ring and γ-Lie ideal of M and prove that if $U \nsubseteq C_\gamma$, $char$M$\neq 2$ and $d^3 \neq 0$, then the γ-subring generated by $d(U)$ contains a nonzero ideal of M. We also prove that if $[u, d(u)]_\gamma \in C_\gamma$ for all $u \in U$, then U is contained in the γ-center of M when char$M \neq 2$ or 3. And if $[u, d(u)]_\gamma \in C_\gamma$ for all $u \in U$ and U is also a γ-subring, then U is γ-commutative when char$M = 2$.

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1. Preliminaries

Let M and Γ be additive Abelian groups. M is said to be a Γ-ring in the sense of Barnes[2] if there exists a mapping $M \times \Gamma \times M \to M$ satisfying these two conditions for all $a, b, c \in M$, $\alpha, \beta \in \Gamma$:

(1) $(a + b)\alpha c = a\alpha c + b\alpha c$
$a(\alpha + \beta)c = a\alpha c + a\beta c$
$a\alpha(b + c) = a\alpha b + a\alpha c$

(2) $(a\alpha b)\beta c = a\alpha(b\beta c)$

In addition, if there exists a mapping $\Gamma \times M \times \Gamma \to \Gamma$ such that the following axioms hold for all $a, b, c \in M$, $\alpha, \beta \in \Gamma$:

(3) $(a\alpha b)\beta c = a(\alpha b\beta)c$

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(4) $a\alpha b = 0$ for all $a, b \in M$ implies $\alpha = 0$, where $\alpha \in \Gamma$

then M is called a Γ-ring in the sense of Nobusawa[10]. If a Γ-ring M in the sense of Barnes satisfies only the condition (3), then it is called weak Nobusawa Γ-ring[9].

Let M be a Γ-ring in the sense of Barnes. Then M is said to be a prime gamma ring if $a\Gamma M\Gamma b = 0$ with $a, b \in M$ implies either $a = 0$ or $b = 0$[2]. It is also defined in [2] that M is a completely prime gamma ring if $a\Gamma b = 0$ with $a, b \in M$ implies either $a = 0$ or $b = 0$.

For a subset U of M and $\gamma \in \Gamma$, the set $C_\gamma(U) = \{a \in M \mid a\gamma u = u\gamma a, \forall u \in U\}$ and the set $C_\gamma = \{a \in M \mid a\gamma m = m\gamma a, \forall m \in M\}$ are called γ-center of the subset U and γ-center of M respectively.

In 2000, Kandamar[7] firstly introduced the notion of a k-derivation for a gamma ring in the sense of Barnes and proved some of its properties and commutativity conditions for Nobusawa gamma rings.

Commutativity conditions with derivations for a gamma ring has been investigated by a number of authors. In [8], Khan, Chaudhry and Javaid proved that if M is a prime gamma ring (in the sense of Barnes) of characteristic not 2, I is a nonzero ideal of M and f is a generalized derivation on M, then M is a commutative gamma ring. In [12], Suliman and Majeed showed a nonzero Lie ideal of a 2-torsion-free prime Γ-ring M with a nonzero derivation d is central if $d(U)$ is contained in the center of M.

In this paper, we define γ-Lie ideal for a weak Nobusawa gamma ring and show that if $U \nsubseteq C_\gamma$, $char$M$\neq 2$ and $d^3 \neq 0$, then the γ-subring generated by $d(U)$ contains a nonzero ideal of M. We also prove that if $[u, d(u)]_\gamma \in C_\gamma$ for all $u \in U$, then $U \subseteq C_\gamma$ when char$M \neq 2$ or 3. And if $[u, d(u)]_\gamma \in C_\gamma$ for all $u \in U$ and U is also a γ-subring, then U is γ-commutative when char$M = 2$.

2. γ-Lie ideals and derivations

Now we give some new definitions and make some preliminary remarks we need later.

Let M be a weak Nobusawa Γ-ring and $0 \neq \gamma \in \Gamma$. A subgroup I of M is said to be a γ-subring if $x\gamma y \in I$ for all $x, y \in I$. A subgroup A of M is said to be a γ-left ideal(resp. γ-right ideal) if $m\gamma a \in A$(resp. $a\gamma m \in A$) for all $m \in M$, $a \in A$. If A is both γ-left and γ-right ideal then A is called a γ-ideal of M.

M is called γ-commutative gamma ring if $x\gamma y = y\gamma x$ for all $x, y \in M$.

We say that the additive subgroup U of M is said to be a γ-Lie ideal of M if $[U, M]_\gamma \subseteq U$. We also say that if there exists a $\gamma \in \Gamma$ such that $a\gamma M\gamma b = 0$ with $a, b \in M$ implies either $a = 0$ or $b = 0$ then M is called a γ-prime gamma ring.

An element a of M is called γ-nilpotent if there exists a positive integer n such that $a_\gamma^n := (a\gamma)^n a = 0$.

In what follows, let M be a γ-prime weak Nobusawa Γ-ring of characteristic not 2, $d \neq 0$ be a k-derivation of M such that $k(\gamma) = 0$ and U be a γ-Lie ideal of M unless otherwise specified.

Lemma 2.1. *If $a \in M$ γ-commutes with $[a, x]_\gamma$ for all $x \in M$, then a is in the γ-center of M.*

Proof. Let $x, y \in M$. Therefore, we get $[a, x]_\gamma \gamma [a, y]_\gamma = 0$ by hypothesis. Replacing y by $m\gamma x$ with $m \in M$, we obtain $[a, x]_\gamma \gamma M\gamma [a, x]_\gamma = 0$. Hence, a is in the γ-center of M since M is γ-prime. □

Lemma 2.2. *Suppose that $U \neq (0)$ is both a γ-subring and a γ-Lie ideal of M. Then either $U \subseteq C_\gamma$ or U contains a nonzero ideal of M.*

Proof. First, suppose that the γ-subring U is not γ-commutative. Then, there exists $x, y \in U$ such that $[x, y]_\gamma \neq 0$. Since U is a γ-Lie ideal, $[x, y]_\gamma \gamma M \subseteq U$. Hence, $\left[[x, y]_\gamma \gamma a, b\right]_\gamma \in U$ for all $a, b \in M$. Expanding this, we get $b\gamma [x, y]_\gamma \gamma a \in U$ leading to $M\gamma [x, y]_\gamma \gamma M \subseteq U$. Moreover, $M\gamma [x, y]_\gamma \gamma M \neq 0$. We have shown that the result is correct if the γ-subring U is not commutative.

Now suppose that U is γ-commutative. Then $\left[a,[a,x]_\gamma\right]_\gamma=0$ for $a\in U$ and $x\in M$. Therefore, we have $U\subseteq C_\gamma$ by Lemma 2.1. □

Lemma 2.3. *Let U be a γ-Lie ideal of M and $U\nsubseteq C_\gamma$. Then there exists a nonzero ideal K of M such that $[K,M]_\gamma\subset U$ but $[K,M]_\gamma\nsubseteq C_\gamma$.*

Proof. Since $U\nsubseteq C_\gamma$, it follows from Lemma 2.1 that $[U,U]_\gamma\neq 0$. Let $K=M\gamma[U,U]_\gamma\gamma M$. Then it is clear that K is a nonzero ideal of M.

Let $T(U)=\left\{x\in M:[x,M]_\gamma\subseteq U\right\}$. Then, it can be shown that $U\subseteq T(U)$ and $T(U)$ is both a γ-subring and a γ-Lie ideal of M. Let $u,v\in U$ such that $[u,v]_\gamma\neq 0$. Replacing v by $v\gamma m$ with $m\in M$, we obtain $[u,v]_\gamma\gamma M\subseteq T(U)$. Hence, $\left[[u,v]_\gamma\gamma m,n\right]_\gamma\in T(U)$ for all $m,n\in M$. Expanding this, we get $K\subseteq T(U)$. Therefore, we have shown that $[K,M]_\gamma\subseteq U$.

Suppose that $[K,M]_\gamma\subseteq C_\gamma$. Then, $\left[x,[x,m]_\gamma\right]_\gamma=0$ for all $x\in K$, $m\in M$. Let $y\in M$. Since $K\subseteq C_\gamma$ by Lemma 2.1, we have $[y,n\gamma k\gamma m]_\gamma=0$ for all $m,n\in M,k\in K$ which leads to $y\in C_\gamma$. But this contradicts with $U\nsubseteq C_\gamma$. □

Lemma 2.4. *Let $u\in M$. If $a\in C_\gamma$ and $a\gamma u\in C_\gamma$, then $a=0$ or $u\in C_\gamma$.*

Proof. Suppose that $a\neq 0$. Since $[a\gamma u,m]_\gamma=0$ for all $m\in M$, we get $a\gamma[u,m]_\gamma=0$. Replacing m by $m\gamma n$ with $n\in M$, we obtain $[u,n]_\gamma=0$ for all $n\in M$. This gives that $u\in C_\gamma$. □

Lemma 2.5. *If U is a γ-Lie ideal of M and $U\nsubseteq C_\gamma$, then $C_\gamma(U)=C_\gamma$.*

Proof. It is clear that $C_\gamma(U)$ is both a γ-subring and γ-Lie ideal of M. We claim that $C_\gamma(U)$ cannot contain a nonzero ideal of M. Suppose K is a nonzero ideal of M which is contained in $C_\gamma(U)$. Then, it is clear that $[u,k\gamma m]_\gamma=0$ for all $u\in U$, $k\in K$ and $m\in M$. Expanding this, we get $k\gamma[u,m]_\gamma=0$. Replacing k by $k\gamma m$ with $m\in M$, we obtain $u\in C_\gamma$ which leads to a contradiction. Hence, $C_\gamma(U)\subseteq C_\gamma$ by Lemma 2.2. □

Lemma 2.6. *If U is a γ-Lie ideal of M, then $C_\gamma\left([U,U]_\gamma\right)=C_\gamma(U)$.*

Proof. First, suppose that $[U,U]_\gamma\nsubseteq C_\gamma$. Since $[U,U]_\gamma$ is a γ-Lie ideal of M, we have $C_\gamma\left([U,U]_\gamma\right)=C_\gamma$ by Lemma 2.5. Now, suppose that $[U,U]_\gamma\subseteq C_\gamma$. Let $a=\left[u,[u,x]_\gamma\right]_\gamma$ for $u\in U$ and $x\in M$. Since $a\in C_\gamma$ and $a\gamma u\in C_\gamma$, we write $a=0$ or $u\in C_\gamma$ by Lemma 2.4. If $a=0$, we have $u\in C_\gamma$ by Lemma 2.1. Hence, we get $U\subseteq C_\gamma$. Thus, $C_\gamma\left([U,U]_\gamma\right)=C_\gamma(U)$. □

Lemma 2.7. *Let U be a γ-Lie ideal of M and $U\nsubseteq C_\gamma$. If $a\gamma U\gamma b=0$ for $a,b\in M$, then $a=0$ or $b=0$.*

Proof. There exists a nonzero ideal K of M such that $[K,M]_\gamma\subset U$ but $[K,M]_\gamma\nsubseteq C_\gamma$ by Lemma 2.3. Thus, $a\gamma[k\gamma a\gamma u,m]_\gamma\gamma b=0$ for $u\in U$, $k\in K$ and $m\in M$ by hypothesis. Expanding this, we get $a\gamma K\gamma a\gamma M\gamma U\gamma b=0$. Since M is γ-prime, we obtain $a\gamma K\gamma a=0$ or $U\gamma b=0$. Let $a\gamma K\gamma a=0$. If $a\neq 0$, then we have $K=0$ which is a contradiction. Now, let $U\gamma b=0$. Therefore, $[u,m]_\gamma\gamma b=0$ for all $u\in U$ and $m\in M$. Hence, we get $U\gamma M\gamma b=0$ which means $b=0$. □

Lemma 2.8. *If d is a k-derivation of M such that $k(\gamma)=0$ and $d^2=0$, then $d=0$.*

Proof. Since $d^2(x\gamma y)=0$ for all $x,y\in M$, we have $d(x)\gamma d(y)=0$. Replacing y by $m\gamma x$ with $m\in M$, we get $d(x)\gamma M\gamma d(x)=0$ for all $x\in M$. Thus, $d=0$ since M is γ-prime. □

Lemma 2.9. *If $d \neq 0$ is a k-derivation of M such that $k(\gamma) = 0$, then $C_\gamma(d(M)) = C_\gamma$.*

Proof. Let $a \in C_\gamma(d(M))$ and suppose $a \notin C_\gamma$. Thus, $[a, d(x\gamma y)]_\gamma = 0$ for all $x, y \in M$. Expanding this, we get $d(x)\gamma[a,y]_\gamma + [a,x]_\gamma \gamma d(y) = 0$. If $y \in M$ γ-commutes with a, then $[a,y]_\gamma = 0$. So the last equation reduces to $[a,x]_\gamma \gamma d(y) = 0$ for all $x \in M$. Then, $d(y) = 0$ since $a \notin C_\gamma$. Indeed, if $d(y) \neq 0$, we get $a \in C_\gamma$. But this is a contradiction. Therefore, $d(y) = 0$ for all $y \in C_\gamma(a)$. Thus, $d = 0$ by Lemma 2.8 which contradicts with the assumption. □

Lemma 2.10. *Let $d \neq 0$ be a k-derivation of M such that $k(\gamma) = 0$ and U be a γ-Lie ideal of M.*

(i) If $d(U) = 0$, then $U \subseteq C_\gamma$.

(ii) If $d(U) \subseteq C_\gamma$ then $U \subseteq C_\gamma$.

Proof. (i) Let $u \in U$ and $x \in M$. Since $d(u) = d\left([u,x]_\gamma\right) = 0$ by hypothesis, we get $[u, d(x)]_\gamma = 0$ for all $x \in M$. Therefore, u centralizes $d(M)$. Then, we get $U \subseteq C_\gamma$ by Lemma 2.9.

(ii) Suppose that $U \nsubseteq C_\gamma$. Then, $V = [U,U]_\gamma \nsubseteq C_\gamma$ by proof of Lemma 2.6. Since $d\left([u,v]_\gamma\right) = 0$ for all $u, v \in U$, we get $d(V) = 0$. It follows that $V \subseteq C_\gamma$ by (i). But this is a contradiction. □

Lemma 2.11. *Let d be a k-derivation of M such that $k(\gamma) = 0$ and U be a γ-Lie ideal of M such that $U \nsubseteq C_\gamma$. If $t\gamma d(U) = 0$ (or $d(U)\gamma t = 0$) for $t \in M$, then $t = 0$.*

Proof. Let $u \in U$ and $x \in M$. Using the fact $[u,x]_\gamma \gamma u = [u, x\gamma u]_\gamma \in U$, we have $t\gamma d\left([u,x]_\gamma \gamma u\right) = 0$. Expanding this, we get $t\gamma [u,x]_\gamma \gamma d(u) = 0$ for all $x \in M$ and $u \in U$. Replacing x by $d(v)\gamma y$ with $v \in U$, $y \in M$, we obtain $t\gamma u\gamma d(v) = 0$ for all $v, u \in U$ since $t\gamma d(U) = 0$ and M is γ-prime. Hence, $t = 0$ by Lemma 2.7. □

Theorem 2.12. *Let $d \neq 0$ be a k-derivation of M such that $k(\gamma) = 0$. If U is a γ-Lie ideal of M such that $d^2(U) = 0$, then $U \subseteq C_\gamma$.*

Proof. Suppose that $U \nsubseteq C_\gamma$. By proof of Lemma 2.6, we have $V = [U,U]_\gamma \nsubseteq C_\gamma$. There exists a nonzero ideal K of M such that $[K,M]_\gamma \subset U$ but $[K,M]_\gamma \nsubseteq C_\gamma$ by Lemma 2.3. Let $y \in M$, $t \in [K,M]_\gamma$ and $u \in V$. If $w := d(u)$, then $d(w) = 0$. By hypothesis, $d^2\left([t\gamma w, y]_\gamma\right) = 0$. Expanding this, we have $d(t)\gamma d\left([w,y]_\gamma\right) = 0$ for all $t \in [K,M]_\gamma$, $y \in M$, $w \in d(V)$. Since $[K,M]_\gamma$ is a γ-Lie ideal of M and $[K,M]_\gamma \nsubseteq C_\gamma$, we have $d\left([d(V), M]_\gamma\right) = 0$ by Lemma 2.11. Expanding last equation, we conclude that $[d(u), d(x)]_\gamma = 0$ for all $x \in M, u \in V$ which means $d(V) \subseteq C_\gamma(d(M))$. Therefore, we have $V \subseteq C_\gamma$ by Lemma 2.9 and by Lemma 2.10. But this is a contradiction. □

Theorem 2.13. *Let $d \neq 0$ be a k-derivation of M such that $k(\gamma) = 0$. If U is a γ-Lie ideal of M such that $U \nsubseteq C_\gamma$, then $C_\gamma(d(U)) = C_\gamma$.*

Proof. Let $a \in C_\gamma(d(U))$ and suppose that $a \notin C_\gamma$. We have $V = [U,U]_\gamma \nsubseteq C_\gamma$ by proof of Lemma 2.6. Since $d(V) \subseteq U$ and $a \in C_\gamma(d(U))$ we get $a\gamma d^2(u) = d^2(u)\gamma a$ and $a\gamma d(u) = d(u)\gamma a$. Now, applying given derivation d to last equation gives $d(a) \in C_\gamma(d(V))$. Since $a \in C_\gamma(d(U))$, $u \in V$ and V is a γ-Lie ideal, we have $[d(a), u]_\gamma = d\left([a,u]_\gamma\right) \in d(V)$. It follows that $[d(a), V]_\gamma = 0$ which means $d(a) \in C_\gamma(V)$. Therefore, $d(a) \in C_\gamma(V) = C_\gamma$ by Lemma 2.5.

Using same process for the element $a\gamma a$ gives $d(a\gamma a) = 2a\gamma d(a) \in C_\gamma$ since $a\gamma a \in C_\gamma(d(U))$. Thus, $d(a) = 0$ by Lemma 2.4. Therefore, if $d(b) \neq 0$ for any $b \in C_\gamma(d(U))$ we have $b \in C_\gamma$. So we get

$a+b\in C_\gamma$ since $d(a+b)=d(b)\neq 0$. Then we have $a\in C_\gamma$. But this is a contradiction. Consequently, when we suppose $C_\gamma(d(U))\nsubseteq C_\gamma$, we are forced to $d(a)=0$ for all $a\in C_\gamma(d(U))$.

Let $W=\{x\in M \mid d(x)=0\}$. Then we have $C_\gamma(d(U))\subseteq W$. Moreover, $d\left([a,u]_\gamma\right)=0$ for any $a\in C_\gamma(d(U))$ and $u\in U$.

There exists a nonzero ideal K of M such that $[K,M]_\gamma\subset U$ but $[K,M]_\gamma\nsubseteq C_\gamma$ by Lemma 2.3. If $t\in[K,M]_\gamma\subset U\cap K$, then $t\gamma a\in K$. Thus, $\left[a,d\left([t\gamma a,u]_\gamma\right)\right]_\gamma=0$. Expanding this, we get $d(t)\gamma\left[a,[a,u]_\gamma\right]_\gamma=0$ for all $t\in[K,M]_\gamma$, $u\in U$. Hence, $\left[a,[a,U]_\gamma\right]_\gamma=0$ by Lemma 2.11. Since $U\nsubseteq C_\gamma$, we have $a\in C_\gamma(U)$. Therefore, $a\in C_\gamma$ by Lemma 2.5. But this is a contradiction. □

Lemma 2.14. *If $d^3\neq 0$ and $U\nsubseteq C_\gamma$, then $\overline{d(M)}$ the γ-subring generated by $d(M)$ contains a nonzero γ-ideal of M.*

Proof. Since $d^2(d(M))\neq 0$, we have $y\in d(M)\subseteq\overline{d(M)}$ such that $d^2(y)\neq 0$. Thus, we get $M\gamma d(y)\subseteq\overline{d(M)}$ since $d(x\gamma y)$ and $d(x)\gamma y$ in $\overline{d(M)}$ for all $x\in M$. Similarly, $d(y)\gamma M\subseteq\overline{d(M)}$. If we act d to the element $a\gamma d(y)\gamma b$ for $a,b\in M$, we get $a\gamma d^2(y)\gamma b\in\overline{d(M)}$ by above, that is to say $M\gamma d^2(y)\gamma M\subseteq\overline{d(M)}$. We also have $M\gamma d^2(y)\subseteq\overline{d(M)}$ and $d^2(y)\gamma M\subseteq\overline{d(M)}$ by above. Therefore, the γ-ideal of M generated by $d^2(y)\neq 0$ contained in $\overline{d(M)}$. □

Lemma 2.15. *Let $d^3\neq 0$, $U\nsubseteq C_\gamma$ and $V=[U,U]_\gamma$. If $\overline{d(V)}$ contains a nonzero left ideal λ of M and a nonzero right ideal ρ of M, then $\overline{d(U)}$ contains a nonzero ideal of M.*

Proof. Since $d(V)\subseteq U$, we have $\overline{d(d(V))}\subseteq\overline{d(U)}$. Let $a\in\lambda\subseteq\overline{d(V)}$ and $x\in M$. Thus, $d(x\gamma a)\in\overline{d(U)}$. Expanding this, we get $x\gamma d(a)\in\overline{d(U)}$ for all $x\in M$ and $a\in\lambda$. Therefore, we have $M\gamma d(\lambda)\subseteq\overline{d(U)}$. Similarly, $d(\rho)\gamma M\subseteq\overline{d(U)}$. Let $a\in\lambda$ and $u\in V$. If we act d to the element $[a,u]_\gamma$, we get $d(a)\gamma u\in\overline{d(U)}$ by above, that is to say $d(\lambda)\gamma V\subseteq\overline{d(U)}$. Similarly, $V\gamma d(\rho)\subseteq\overline{d(U)}$.

Let $I=\lambda\gamma V\gamma\rho$. Then by Lemma 2.7, I is a nonzero ideal of M. Moreover, $\overline{d(I)}\subseteq\overline{d(U)}$. By Lemma 2.14, $\overline{d(I)}$ contains a nonzero γ-ideal K of I since $d^3\neq 0$ and I is γ-prime. Let $S:=\lambda\gamma K\gamma\rho$. Then S is an ideal of M which is contained in $\overline{d(U)}$. □

Lemma 2.16. *Let $0\neq I<M$ and $U\nsubseteq C_\gamma$. If $\overline{d(U)}$ does not contain a nonzero right ideal(or left ideal) of M and $[c,I]_\gamma\subseteq\overline{d(U)}$ then $c\in C_\gamma$.*

Proof. Let $t\in d(U)$ and $i\in I$. Then $[c,t\gamma i]_\gamma\in\overline{d(U)}$ by hypothesis. Expanding this, we get $[c,d(U)]_\gamma\gamma I\subseteq\overline{d(U)}$. But, since $\overline{d(U)}$ does not contain a nonzero right ideal of M, we get $[c,d(U)]_\gamma\gamma I=0$. Thus, $[c,d(U)]_\gamma=0$ since $0\neq I<M$ and M is a γ-prime gamma ring. Then by Theorem 2.13, we get $c\in C_\gamma(d(U))=C_\gamma$. □

Lemma 2.17. *Let $U\nsubseteq C_\gamma$, $V=[U,U]_\gamma$ and $W=[V,V]_\gamma$. If $d^2(U)\gamma d^2(U)=0$ then $d^3(W)=0$.*

Proof. By proof of Lemma 2.6, V and W are not contained in C_γ since $U\nsubseteq C_\gamma$. Also, we have $d(W)\subseteq V$ and $d^2(W)\subseteq d(V)\subseteq U$. If $u\in U$, $v\in V$ and $w\in W$, then we have $d^2(u)\gamma d^2\left(\left[d(v),d^2(w)\gamma t\right]_\gamma\right)=0$ for any $t\in U$. Expanding this, we get

$$d^2(u)\gamma d(v)\gamma\left(d^4(w)\gamma t+2d^3(w)\gamma d(t)\right)=0 \tag{1}$$

by hypothesis. In (1), if we choose $t\in d(V)\subseteq U$, it follows $d^2(u)\gamma d(v)\gamma d^4(w)\gamma t=0$ for such t. Thus, we have $d^2(u)\gamma d(v)\gamma d^4(w)=0$ by Lemma 2.11. Then we get from (1) that $d^2(u)\gamma d(v)\gamma d^3(w)\gamma d(t)=$

0 for all $t \in U$. By Lemma 2.11, we conclude that $d^2(u)\gamma d(v)\gamma d^3(w) = 0$ for all $u \in U$, $v \in V$ and $w \in W$. Similarly, we have $d^3(w)\gamma d(v)\gamma d^2(u) = 0$ by reversing sides.

By hypothesis, $d^2(d(t))\gamma d^2\left([v, d(w)]_\gamma\right) = 0$ for $w, t \in W$ and $v \in V$. Expanding this, we get $d^3(t)\gamma v\gamma d^3(w) = 0$ that is to say $d^3(t)\gamma V\gamma d^3(w) = 0$ for all $w, t \in W$. It follows that $d^3(W) = 0$ by Lemma 2.7. □

Lemma 2.18. *If $U \nsubseteq C_\gamma$ and $d^3(U) = 0$ then $d^3 = 0$.*

Proof. Let $u \in U$ and $m \in M$. Then we have $d^3\left([u, m]_\gamma\right) = 0$. Expanding this, we get

$$3\left[d^2(u), d(m)\right]_\gamma + 3\left[d(u), d^2(m)\right]_\gamma + \left[u, d^3(m)\right]_\gamma = 0. \tag{2}$$

Let $V = [U, U]_\gamma$ and $W = [V, V]_\gamma$. In (2), replacing u by $d^2(w)$ with $w \in W$, we get $\left[d^2(w), d^3(m)\right]_\gamma = 0$ by hypothesis. Again replacing u by $d(w)$ and m by $d(m)$ in (2), we obtain $\left[d(w), d^4(m)\right]_\gamma = 0$.

By proof of Lemma 2.6, $W \nsubseteq C_\gamma$. Thus, by Theorem 2.13, $C_\gamma(d(W)) = C_\gamma$. Since $\left[d(w), d^4(m)\right]_\gamma = 0$ for all $w \in W$ and $m \in M$, it follows $d^4(M) \subseteq C_\gamma$.

By hypothesis, $d^4\left([u, m]_\gamma\right) = 0$ for $u \in U$ and $m \in M$. Expanding this, we get

$$6\left[d^2(u), d^2(m)\right]_\gamma + 4\left[d(u), d^3(m)\right]_\gamma = 0 \tag{3}$$

since $d^4(m) \in C_\gamma$. Similarly, expanding the equation $d^3\left([u, d(m)]_\gamma\right) = 0$, we get

$$3\left[d^2(u), d^2(m)\right]_\gamma + 3\left[d(u), d^3(m)\right]_\gamma = 0. \tag{4}$$

Combining the equation (4) and the equation (3) we get $\left[d(u), d^3(m)\right]_\gamma = 0$ for all $u \in U$ and $m \in M$. Therefore, by Theorem 2.13, $d^3(M) \subseteq C_\gamma(d(U)) = C_\gamma$. Hence, we get $d^3(m)\gamma d^2(u) \in C_\gamma$ that is to say $d^3(M)\gamma d^2(U) \subseteq C_\gamma$.

Suppose that $d^3(M) \neq 0$. By Lemma 2.4, we have $d^2(U) \subseteq C_\gamma$. Since $d^4(m\gamma d(u)) \in C_\gamma$ it follows $d^4(M)\gamma d(U) \subseteq C_\gamma$. Since $d(U)$ cannot be contained in C_γ by Lemma 2.10, we get $d^4(M) = 0$ by Lemma 2.4. So $d^4(m\gamma d(u)) = 4d^3(m)\gamma d^2(u) = 0$. Hence, $d^3(M)\gamma d^2(U) = 0$. On the other hand, we have $d^2(U) \neq 0$ by Theorem 2.12. Since $d^2(U) \subseteq C_\gamma$ and M is γ-prime gamma ring, $d^3(M) = 0$, that is to say $d^3 = 0$. □

Lemma 2.19. *If $[U, U]_\gamma \subseteq C_\gamma$ then $U \subseteq C_\gamma$.*

Proof. Let $[U, U]_\gamma = 0$. Then, we get $\left[u, [u, x]_\gamma\right]_\gamma = 0$ for all $u \in U$ and $x \in M$ by hypothesis. Therefore, $u \in C_\gamma$ by Lemma 2.1.

Now, let $[U, U]_\gamma \neq 0$. Then, there exist $u, v \in U$ such that $[u, v]_\gamma \neq 0$. Let $d(x) = [x, v]_\gamma$ for $x \in M$. Then $d^2(x) = [d(x), v]_\gamma \in C_\gamma$ for all $x \in M$ by hypothesis. Let $a = d(u)$ and $b = d^2(x)$. Therefore, $d^2(u\gamma x) = 2a\gamma d(x) + b\gamma u \in C_\gamma$. Then we have $[u, 2a\gamma d(x) + b\gamma u]_\gamma = 0$. Expanding this, we get $2a\gamma[u, d(x)]_\gamma = 0$ for all $x \in M$. Replacing x by $u\gamma v$ in the last equation, we obtain $a_\gamma^3 = 0$. Therefore, we have a nonzero γ-nilpotent element a in the γ-center of the γ-prime gamma ring M. But this is a contradiction. □

Lemma 2.20. *Let M be a gamma ring of characteristic not 2 and U be a γ-Lie ideal of M. If $[u, d(u)]_\gamma \in C_\gamma$ and $u^2 \in U$ for all $u \in U$, then $[u, d(u)]_\gamma = 0$.*

Proof. We know that $\left[u+u^2, d(u+u^2)\right]_\gamma \in C_\gamma$ for all $u \in U$ by hypothesis. Expanding this, we get $4\,[u,d(u)]_\gamma\,\gamma u \in C_\gamma$. Hence, $[u,d(u)]_\gamma\,\gamma\,[u,m]_\gamma = 0$ for all $u \in U$ and $m \in M$. Replacing m by $m\gamma x$ with $x \in M$, we obtain $[u,d(u)]_\gamma\,\gamma m\gamma\,[u,x]_\gamma = 0$ that leads to $[u,d(u)]_\gamma = 0$ or $[u,x]_\gamma = 0$ for all $x \in M$ since M is γ-prime gamma ring. ☐

Lemma 2.21. *Let U be a γ-Lie ideal of M and $[u,d(u)]_\gamma \in C_\gamma$ for all $u \in U$. Then $\left[[d(m),u]_\gamma, u\right]_\gamma \in C_\gamma$ for all $u \in U$ and $m \in M$. Moreover, if $[u,d(u)]_\gamma = 0$ for all $u \in U$, then $\left[[d(m),u]_\gamma, u\right]_\gamma = 0$ for all $u \in U$ and $m \in M$.*

Proof. Let $u \in U$ and $m \in M$. By hypothesis, $\left[u+[u,m]_\gamma, d(u+[u,m]_\gamma)\right]_\gamma \in C_\gamma$. Expanding this, we get

$$\left[[u,m]_\gamma, d(u)\right]_\gamma + \left[u,[d(u),m]_\gamma\right]_\gamma + \left[u,[u,d(m)]_\gamma\right]_\gamma \in C_\gamma.$$

But for any $u \in U$ and $m \in M$ one can show that

$$\left[[u,m]_\gamma, d(u)\right]_\gamma + \left[u,[d(u),m]_\gamma\right]_\gamma = \left[m,[d(u),u]_\gamma\right]_\gamma = 0.$$

Therefore, we get desired result. Similary, the other statement can easily be shown. ☐

Lemma 2.22. *Let $a \in M$. If $a\gamma d(x) = 0$ for all $x \in M$, then $a = 0$ or $d = 0$.*

Proof. By hypothesis, $a\gamma d(x\gamma y) = 0$ for all $x, y \in M$. Expanding this, we get $a\gamma x\gamma d(y) = 0$ for all $x, y \in M$. Since M is γ-prime gamma ring we have the desired result. ☐

3. Main results

Theorem 3.1. *Let M be a γ-prime weak Nobusawa Γ-ring of characteristic not 2 and d be a k-derivation of M such that $k(\gamma) = 0$ and $d^3 \neq 0$. If U is a γ-Lie ideal of M such that $U \nsubseteq C_\gamma$ then $\overline{d(U)}$ contains a nonzero ideal of M.*

Proof. Let $V = [U,U]_\gamma$ and $W = [V,V]_\gamma$. According to Lemma 2.15, it is enough to show that the γ-subring $\overline{d(V)}$ contains a nonzero left ideal of M and a nonzero right ideal of M. Suppose that $\overline{d(V)}$ does not contain a nonzero right ideal of M.

Let $w \in [W,W]_\gamma$ and $a := d(w)$. Since $a\gamma\,[a,x]_\gamma \in W$, we have $d\left(a\gamma\,[a,x]_\gamma\right) \in d(W)$. Expanding this, we get

$$d(a)\,\gamma\,[a,x]_\gamma \in \overline{d(V)},\ \forall a \in d\left([W,W]_\gamma\right),\ x \in M. \tag{5}$$

On the other hand, since $d\left([a,u]_\gamma\right) \in d(V)$ and $[a,d(u)]_\gamma \in \overline{d(V)}$ for $u \in V$, we have

$$[d(a), V]_\gamma \subseteq \overline{d(V)},\ \forall a \in d\left([W,W]_\gamma\right). \tag{6}$$

For $m \in M$ we also have

$$d(a)\,\gamma\,[d(a),m]_\gamma + d(a)\,\gamma\,[a,d(m)]_\gamma \in \overline{d(V)}$$

since $d(a)\gamma d\left([a,m]_\gamma\right) \in \overline{d(V)}$. Hence, by (5)

$$d(a)\gamma[d(a),m]_\gamma \in \overline{d(V)},\ \forall a \in d\left([W,W]_\gamma\right),\ m \in M. \tag{7}$$

In (7) replacing a by $a+b$ with $a,\ b \in d\left([W,W]_\gamma\right)$ we obtain

$$s := d(a)\gamma[d(b),m]_\gamma + d(b)\gamma[d(a),m]_\gamma \in \overline{d(V)},\ \forall a,b \in d\left([W,W]_\gamma\right).$$

If $t := [d(a)\gamma d(b),m]_\gamma = d(a)\gamma[d(b),m]_\gamma + [d(a),m]_\gamma \gamma d(b)$ then

$$s - t = d(b)\gamma[d(a),m]_\gamma - [d(a),m]_\gamma \gamma d(b) = \left[d(b),[d(a),m]_\gamma\right]_\gamma.$$

By (6), $s-t \in \overline{d(V)}$. Thus, we get $t \in \overline{d(V)}$, that is $[d(a)\gamma d(b),M]_\gamma \subseteq \overline{d(V)}$. Since $\overline{d(V)}$ does not contain a nonzero right ideal of M, $d(a)\gamma d(b) \in C_\gamma$ for all $a,b \in d\left([W,W]_\gamma\right)$ by Lemma 2.16. Let $n := d(a)\gamma d(b)$. By (5), $d(b)\gamma[b,x]_\gamma \in \overline{d(V)}$. It follows $n\gamma[b,x]_\gamma = d(a)\gamma d(b)\gamma[b,x]_\gamma \in \overline{d(V)}$ since $d(a) \in \overline{d(V)}$. On the other hand, since $n\gamma[b,x]_\gamma = [b,n\gamma x]_\gamma \in \overline{d(V)}$, we have $[b,n\gamma M]_\gamma \subseteq \overline{d(V)}$. Let $I = n\gamma M$. If $I \neq 0$, then $b \in C_\gamma$ for all $b \in d\left([W,W]_\gamma\right)$ by Lemma 2.16. Thus, by Lemma 2.10 we get $[W,W]_\gamma \subseteq C_\gamma$ that is to say $U \subseteq C_\gamma$ by Lemma 2.19. But this is a contradiction. Therefore, $I = n\gamma M = 0$. So we get $n = d(a)\gamma d(b) = 0$ for all $a,b \in d\left([W,W]_\gamma\right)$ since M is γ-prime gamma ring. That is, $d^2\left([W,W]_\gamma\right)\gamma d^2\left([W,W]_\gamma\right) = 0$. Hence, we conclude that the contradiction $d^3 = 0$ by Lemma 2.17 and Lemma 2.18. □

Theorem 3.2. *Let M be a gamma ring of characteristic not 2 or 3 and U be a γ-Lie ideal of M. If $[u,d(u)]_\gamma \in C_\gamma$ for all $u \in U$, then $U \subset C_\gamma$.*

Proof. By Lemma 2.21 we have

$$\left[[d(m),u]_\gamma,u\right]_\gamma \gamma u = u\gamma\left[[d(m),u]_\gamma,u\right]_\gamma.$$

Expanding this, we get

$$3u^2\gamma d(m)\gamma u + d(m)\gamma u^3 = 3u\gamma d(m)\gamma u^2 + u^3\gamma d(m). \tag{8}$$

Let $d(m) = m'$. Replacing m by u in (8), we obtain

$$u^3\gamma u' - u'\gamma u^3 = 3(u\gamma u' - u'\gamma u)\gamma u^2 \tag{9}$$

for all $u \in U$. Since $[u,u']_\gamma \gamma u = u\gamma[u,u']_\gamma$ we have

$$2\gamma(u\gamma u' - u'\gamma u)\gamma u = u^2\gamma u' - u'\gamma u^2. \tag{10}$$

Again replacing m by $u\gamma m'$ in (8), we obtain

$$3u\gamma u'\gamma m'\gamma u^2 + u^3\gamma u'\gamma m' - 3u^2\gamma u'\gamma m'\gamma u - u'\gamma m'\gamma u^3 = 0 \tag{11}$$

for all $u \in U$ and $m \in M$. Multiplying (8) with u' gives

$$3u'\gamma u\gamma m'\gamma u^2 + u'\gamma u^3\gamma m' - 3u'\gamma u^2\gamma m'\gamma u - u'\gamma m'\gamma u^3 = 0.$$

Substracting the last equation from (11) we get

$$3(u\gamma u' - u'\gamma u)\gamma m'\gamma u^2 + (u^3\gamma u' - u'\gamma u^3)\gamma m' - 3(u^2\gamma u' - u'\gamma u^2)\gamma m'\gamma u = 0.$$

Using the equations (9) and (10) we conclude

$$(u\gamma u' - u'\gamma u)\gamma(m'\gamma u^2 + u^2\gamma m' - 2u\gamma m'\gamma u) = 0$$

for all $u \in U$ and $m \in M$. If $u\gamma u' - u'\gamma u \neq 0$ for some u, then

$$m'\gamma u^2 + u^2\gamma m' - 2u\gamma m'\gamma u = 0 \tag{12}$$

for that u. Replacing m by $u\gamma m$, we obtain

$$(u'\gamma m + u\gamma m')u^2 + u^2\gamma(u'\gamma m + u\gamma m') - 2u\gamma(u'\gamma m + u\gamma m') = 0.$$

Expanding last equation, we have

$$u'\gamma m\gamma u^2 + u^2\gamma u'\gamma m - 2u\gamma u'\gamma m\gamma u = 0 \tag{13}$$

for all $m \in M$. If we replace m by u in (12) and multiply by m on the right, then we get

$$u'\gamma u^2\gamma m + u^2\gamma u'\gamma m - 2u\gamma u'\gamma u\gamma m = 0. \tag{14}$$

Substracting (14) from (13) gives

$$u'\gamma(m\gamma u^2 - u^2\gamma m) - 2u\gamma u'\gamma(m\gamma u - u\gamma m) = 0. \tag{15}$$

Replacing m by $u\gamma m$ in (15), we obtain

$$u'\gamma u\gamma(m\gamma u^2 - u^2\gamma m) - 2u\gamma u'\gamma u\gamma(m\gamma u - u\gamma m) = 0. \tag{16}$$

Mulyiplying (15) by u we get

$$u\gamma u'\gamma(m\gamma u^2 - u^2\gamma m) - 2u^2\gamma u'\gamma(m\gamma u - u\gamma m) = 0. \tag{17}$$

Substracting (16) from (17) gives

$$(u\gamma u' - u'\gamma u)\gamma(m\gamma u^2 - u^2\gamma m - 2u\gamma(m\gamma u - u\gamma m)) = 0$$

for all $m \in M$. Since M is γ-prime gamma ring we have

$$m\gamma u^2 - u^2\gamma m - 2u\gamma(m\gamma u - u\gamma m) = 0.$$

Now think the inner $I_{\gamma u}$-derivation $I_{u\gamma}$ on M. From the last equation we write $I_{u\gamma}^2 = 0$ that leads to $I_{u\gamma} = 0$ by Lemma 2.8. Hence, $u \in C_\gamma$.

So far we proved that if $[u, u']_\gamma \neq 0$ for $u \in U$, then $u \in C_\gamma$. Now assume that $[u, u']_\gamma = 0$ for all $u \in U$. By Lemma 2.21, $\left[[d(m), u]_\gamma, u\right]_\gamma = 0$ for all $m \in M$ and $u \in U$. Replacing u by $u + w$ with $w \in U$, we obtain

$$\left[[d(m), u]_\gamma, w\right]_\gamma + \left[[d(m), w]_\gamma, u\right]_\gamma = 0. \tag{18}$$

Choose $v, w \in U$ such that $w\gamma v \in U$. Replacing w by $w\gamma v$ in (18), we obtain

$$[w, u]_\gamma \gamma [d(m), v]_\gamma + [d(m), w]_\gamma \gamma [v, u]_\gamma = 0.$$

For any $t \in M$ and $w \in U$ the element $v = [t, w]_\gamma$ ensures the condition $w\gamma v \in U$. So by above we have

$$[w, u]_\gamma \gamma \left[d(m), [t, w]_\gamma\right]_\gamma + [d(m), w]_\gamma \gamma \left[[t, w]_\gamma, u\right]_\gamma = 0 \tag{19}$$

for all $t, m \in M$ and $u, w \in U$. Replacing u by w, we obtain

$$[d(m), w]_\gamma \gamma \left[[t, w]_\gamma , w\right]_\gamma = 0. \tag{20}$$

Replacing t by $t\gamma d(a)$ with $a \in M$, we obtain

$$[d(m), w]_\gamma \gamma [t, w]_\gamma \gamma [d(a), w]_\gamma = 0 \tag{21}$$

for all $m, t, a \in M$ and $w \in U$. Replacing u by $[t, w]_\gamma$ in (19), we get $\left[[t, w]_\gamma , w\right]_\gamma \gamma \left[[t, w]_\gamma , d(m)\right]_\gamma = 0$. If we replace t by $t + d(a)$ we get

$$\left[[t, w]_\gamma , w\right]_\gamma \gamma \left[[d(a), w]_\gamma , d(m)\right]_\gamma = 0 \tag{22}$$

for all $m, t, a \in M$ and $w \in U$. Replacing t by $d(t)\gamma s$ with $s \in M$ in (22), we obtain

$$[d(t), w]_\gamma \gamma [s, w]_\gamma \gamma d(m)\gamma [d(a), w]_\gamma = 0$$

for all $m, t, a, s \in M$ and $w \in U$.

Replacing t by $t\gamma d(s)$ in (21), we conclude

$$[d(m), w]_\gamma \gamma M\gamma [d(s), w]_\gamma \gamma [d(a), w]_\gamma = 0$$

for all $m, a, s \in M$ and $w \in U$. Since M is γ-prime gamma ring we get $[d(m), w]_\gamma = 0$ or $[d(s), w]_\gamma \gamma [d(a), w]_\gamma = 0$ for all $m, a, s \in M$ and $w \in U$. If $[d(m), w]_\gamma = 0$ for all $m \in M$ and $w \in U$, then $w \in C_\gamma$ by Lemma 2.9, so we are done. Suppose there is a pair $m \in M$ and $w \in U$ such that $[d(m), w]_\gamma \neq 0$. Hence, $w \notin C_\gamma$ and

$$[d(s), w]_\gamma \gamma [d(a), w]_\gamma = 0 \tag{23}$$

for all $a, s \in M$. Replacing a by $b\gamma c$ with $b, c \in M$ in (23), we get

$$[d(s), w]_\gamma \gamma d(b)\gamma [c, w]_\gamma = 0.$$

If we replace b by $[t, w]_\gamma$ in this equation we have

$$[d(s), w]_\gamma \gamma [t, d(w)]_\gamma \gamma [w, c]_\gamma = 0$$

for all $c, t, s \in M$ and $w \in U$. Replacing c by $c\gamma m_1$ with $m_1 \in M$, we obtain

$$[d(s), w]_\gamma \gamma [t, d(w)]_\gamma = 0.$$

Hence, replacing t by $t\gamma k$ with $k \in M$ in the last equation, we get $d(w) \in C_\gamma$.

Now suppose $u \in U \cap C_\gamma$. Then $d([u, a]_\gamma) = 0$ for all $a \in M$. Therefore, we have $d(u) \in C_\gamma$. Hence, $d(u) \in C_\gamma$ for all $u \in U$ and then we get $d([w, a]_\gamma) \in C_\gamma$ for all $a \in M$. Expanding this, we obtain $[w, d(a)]_\gamma \in C_\gamma$ and replacing a by $a\gamma w$, we have

$$[w, d(a)]_\gamma \gamma w + [w, a]_\gamma \gamma d(w) \in C_\gamma. \tag{24}$$

Therefore, commuting this element by w we get

$$\left[w, [w, a]_\gamma\right]_\gamma \gamma d(w) = 0$$

for all $a \in M$. Since M is γ-prime gamma ring we have $\left[w, [w, a]_\gamma\right]_\gamma = 0$ or $d(w) = 0$ for all $a \in M$. If $\left[w, [w, a]_\gamma\right]_\gamma = 0$, then $w \in C_\gamma$ by Lemma 2.1. But this is a contradiction. Hence, $d(w) = 0$ and $[w, d(a)]_\gamma \gamma w \in C_\gamma$ for all $a \in M$ by (24). It follows that $[d(a), w]_\gamma \gamma [w, b]_\gamma = 0$ for $a, b \in M$. Replacing b by $b\gamma c$ with c, we obtain $[d(a), w]_\gamma = 0$ or $[w, b]_\gamma = 0$. So in both cases we have $w \in C_\gamma$ which is a contradiction. □

Corollary 3.3. *Let M be a gamma ring of characteristic 3 and U be a γ-Lie ideal of M. If $[u, d(u)]_\gamma \in C_\gamma$ and $u^2 \in U$ for all $u \in U$, then $U \subset C_\gamma$.*

Theorem 3.4. *Let M be a gamma ring of characteristic 2 and U be a γ-Lie ideal and γ-subring of M. If $[u, d(u)]_\gamma \in C_\gamma$ for all $u \in U$, then U is γ-commutative.*

Proof. By Lemma 2.21, $\left[[d(m), u]_\gamma, u\right]_\gamma \in C_\gamma$ for all $u \in U$ and $m \in M$. Hence,

$$d(m)\gamma u^2 + u^2\gamma d(m) \in C_\gamma \tag{25}$$

for all $u \in U$ and $m \in M$. Then

$$\left[d(m), d(m)\gamma u^2 + u^2\gamma d(m)\right]_\gamma = 0$$

and

$$\left[u^2, d(m)\gamma u^2 + u^2\gamma d(m)\right]_\gamma = 0.$$

Expanding these equations, we get

$$u^2\gamma(d(m))^2 = (d(m))^2\gamma u^2 \tag{26}$$

and

$$u^4\gamma d(m) = d(m)\gamma u^4 \tag{27}$$

respectively.

Since $d(u^2) = u\gamma d(u) + d(u)\gamma u \in C_\gamma$ for $u \in U$, replacing m by $v + u^2\gamma v$ with $v \in U$, we obtain

$$\left(u^2\gamma d(v) + d(v)\gamma u^2\right)^2 = 0$$

for all $u, v \in U$. Using γ-primeness of M we have

$$u^2\gamma d(v) = d(v)\gamma u^2 \tag{28}$$

for all $u, v \in U$ by (25).

Replacing u by $u + w$ with $w \in U$ in (28), we get

$$(u\gamma w + w\gamma u)\,\gamma d(v) = d(v)\gamma\,(u\gamma w + w\gamma u)\,.$$

Replacing w by $w\gamma u$, we get

$$(u\gamma w + w\gamma u)\,\gamma\,(u\gamma d(v) + d(v)\gamma u) = 0$$

for all $u, v, w \in U$. We conclude

$$\left(u_1^2\gamma w + w\gamma u_1^2\right)\gamma\,(u\gamma d(u) + d(u)\gamma u) = 0,\ \forall u, u_1, w \in U$$

replacing u by $u + u_1^2$ with $u_1 \in U$ and taking $v = u$.

Hence, if $[d(u), u]_\gamma \neq 0$ for some $u \in U$, then $u_1^2\gamma w = w\gamma u_1^2$ for all $u_1, w \in U$. Then, we have $u^2\gamma\,(w\gamma m + m\gamma w) = (w\gamma m + m\gamma w)\,\gamma u^2$ for all $m \in M$ and $u, w \in U$. Expanding this, and replacing m by $m\gamma u$, we obtain

$$\left(u^2\gamma m + m\gamma u^2\right)\gamma\,(w\gamma u + u\gamma w) = 0$$

for all $u, w \in U$ and $m \in M$. Replacing w by $[u,t]_\gamma$ with $t \in M$, we get

$$\left(u^2\gamma m + m\gamma u^2\right)\gamma\left(u^2\gamma t + t\gamma u^2\right) = 0$$

for all $u \in U$ and $m, t \in M$. Again replacing t by $t\gamma p$ with $p \in P$, we conclude $u^2 \in C_\gamma$ for all $u \in U$.

Assume that $[d(u), u]_\gamma = 0$ for all $u \in U$. Then, by Lemma 2.21 $\left[[d(m), u]_\gamma, u\right]_\gamma = 0$ for all $u \in U$ and $m \in M$. Expanding this, we get $u^2\gamma d(m) = d(m)\gamma u^2$. Replacing m by $m\gamma a$ with $a \in M$, we obtain

$$d(m)\gamma\left(u^2\gamma a + a\gamma u^2\right) + \left(u^2\gamma m + m\gamma u^2\right)\gamma d(a) = 0.$$

Replacing a by v^2, we get $d(m)\gamma\left(u^2\gamma v^2 + v^2\gamma u^2\right) = 0$ for all $u, v \in U, m \in M$ since $d(v^2) = v\gamma d(v) + d(v)\gamma v = 0$ for $v \in U$. Therefore, $u^2\gamma v^2 = v^2\gamma u^2$ for all $u, v \in U$ by Lemma 2.22. Hence, $u^2\gamma\left(v\gamma w + w\gamma v\right) = \left(v\gamma w + w\gamma v\right)\gamma u^2$ for all $u, v, w \in U$. Replacing v by $v\gamma w$ in the last equation, we have $\left(v\gamma w + w\gamma v\right)\gamma\left(u^2\gamma w + w\gamma u^2\right) = 0$. Again replacing v by $[w, m]_\gamma$ with $m \in M$, we obtain $\left(w^2\gamma m + m\gamma w^2\right)\gamma\left(u^2\gamma w + w\gamma u^2\right) = 0$. That is, $I_{w^2\gamma}(m)\gamma\left(u^2\gamma w + w\gamma u^2\right) = 0$ for the inner $I_{\gamma w^2}$-derivation $I_{w^2\gamma}$ on M. So by Lemma 2.22, if $w^2 \notin C_\gamma$ for some $w \in U$, then $u^2\gamma w = w\gamma u^2$ for that w. Therefore, $\left[[u,v]_\gamma, w\right]_\gamma = 0$ for all $u, v \in U$. Expanding last equation and replacing v by $v\gamma w$, we obtain $[v,w]_\gamma\gamma[w,u]_\gamma = 0$ for all $u, v \in U$. Replacing v by $[w,r]_\gamma$ with $m, t \in M$ and u by $[w,t]_\gamma$, we get $\left(w^2\gamma m + m\gamma w^2\right)\left(w^2\gamma t + t\gamma w^2\right) = 0$ for all $m, t \in M$. Again replacing t by $t\gamma p$ with $p \in M$, we conclude $\left(w^2\gamma m + m\gamma w^2\right)\gamma t\gamma\left(w^2\gamma p + p\gamma w^2\right) = 0$ for all $p, t \in M$. Since M is γ-prime gamma ring we get $w^2 \in C_\gamma$ from the last equation. But this is a contradiction.

So far we conclude $u^2 \in C_\gamma$ for all $u \in U$. Hence, $u\gamma v + v\gamma u \in C_\gamma$ and $\left(u\gamma v + v\gamma u\right)\gamma u \in C_\gamma$ for all $u, v \in U$. Therefore, we have $u \in C_\gamma$ or $u\gamma v + v\gamma u = 0$. Then, U is γ-commutative. □

If we assume that U is only γ-Lie ideal of M or only γ-subring of M, then U may not be γ-commutative. Moreover, according to the assumptions of the theorem, the result $U \subseteq C_\gamma$ cannot be obtained.

Example 3.5. *Let R be a noncommutative prime ring with identity. If M is the set of all matrices over R of the form $\begin{pmatrix} a & b & a \\ c & d & c \end{pmatrix}$, $\Gamma = \mathcal{M}_{3\times 2}(R)$ and $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \in \Gamma$, then M is a γ-prime gamma ring. It can be shown that the subset $U = \left\{\begin{pmatrix} a & 0 & a \\ 0 & b & 0 \end{pmatrix} \mid a, b \in R\right\}$ of M is a γ-subring but it is not a γ-Lie ideal of M. Define the inner $I_{\gamma n}$-derivation $I_{n\gamma}$ on M with $n = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \in M$. Then it is easily verified that $[u, I_{n\gamma}(u)]_\gamma \in C_\gamma$ for all $u \in U$ but U is not γ-commutative.*

Example 3.6. *Let $M = \left\{\begin{pmatrix} a & b & a \\ c & d & c \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}_2\right\}$, $\Gamma = \mathcal{M}_{3\times 2}(\mathbb{Z}_2)$ and $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \in \Gamma$. Then M is a γ-prime gamma ring. Let $U = \left\{\begin{pmatrix} a & b & a \\ c & a & c \end{pmatrix} \mid a, b, c \in \mathbb{Z}_2\right\}$. It is easily seen that U is a γ-Lie ideal but it is not a γ-subring of M. Define the inner $I_{\gamma n}$-derivation $I_{n\gamma}$ on M with $n = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \in M$. Then it is easily verified that $[u, I_{n\gamma}(u)]_\gamma \in C_\gamma$ for all $u \in U$ but U is not γ-commutative.*

Example 3.7. *Let $M = \left\{\begin{pmatrix} a & b & a \\ c & d & c \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}_2\right\}$, $\Gamma = \mathcal{M}_{3\times 2}(\mathbb{Z}_2)$ and $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \in \Gamma$. Then M is a γ-prime gamma ring. Let $U = \left\{\begin{pmatrix} a & b & a \\ b & a & b \end{pmatrix} \mid a, b \in \mathbb{Z}_2\right\}$. It is easily seen that U is a γ-Lie* *ideal and a γ-subring of M but it is not a γ-ideal. Define the inner $I_{\gamma n}$-derivation $I_{n\gamma}$ on M with $n = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \in M$. Then it is easily verified that $[u, I_{n\gamma}(u)]_\gamma \in C_\gamma$ for all $u \in U$. Hence, U is γ-commutative but cannot be contained in the γ-center of M.*

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