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Research Article

Conharmonic curvature tensor on nearly cosymplectic manifolds with generalized tanaka-webster connection

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ABSTRACT

In this paper, we study conharmonic curvature tensor of nearly cosymplectic manifolds with generalized Tanaka-Webster connection and we give a conharmonically flat nearly cosymplectic manifold with respect to the connection.

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INTRODUCTION

Recently, many studies have been conducted on nearly structures by different mathematicians in different manifolds. If we talk about some of them, the notion of a nearly Sasakian manifold was introduced by Blair and his collaborators in [5], while nearly cosymplectic manifolds were studied by Blair and Showers in [2]–[4] In the subsequent literature on the topic, the papers on Olszak were quite important [14],[15] for nearly Sasakian manifolds and those of Endo [8],[9] on nearly cosymplectic manifolds.

Later on, these two classes have played a role in the Chinea-Gonzalez's classification of almost contact metric manifolds [7]. They have also appeared in the study of harmonic almost contact structures (cf. [10],[16]). An almost contact metric structure (ϕ,η,ξ,g) satisfying $(\nabla_X \phi)X = 0$ is called a nearly cosymplectic structure. If we consider S^5 as a totally geodesic hypersurface of S^6; then it is known that S^5 has a non cosymplectic nearly cosymplectic structure. It was shown that the normal nearly cosymplectic manifolds are cosymplectic (see [5]). In [12], Loubeau and Vergara-Diaz proved that a nearly cosymplectic structure, once identified with a section of a twistor bundle, always defines a harmonic map. On the other hand, almost contact manifolds with Killing structures tensors were defined in [6] as nearly cosymplectic manifolds. Later on, Blair and Showers [4] studied nearly cosymplectic structure (ϕ,η,ξ,g) on a Riemannian manifold M with η closed from the topological viewpoint.

In addition, Tanaka-Webster connection is canonical affine connection defined on a non-degenerate pseudo-Hermition CR-manifolds. A generalized Tanaka-Webster

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connection has been introduced by Tanno [13] as a generalization of Tabaka-Webster connection [11],[12]. Contact manifolds with generalized Tanaka-Webster connection were studied by many researchers [17]-[21].

The paper is organized as: after introduction section, in Section 2, we give some information about almost contact Riemannian manifolds and accordingly, nearly cosymplectic manifolds and the fundamental curvature properties they provide. In Section 3, we discussed some equations provided by the Riemannian curvature tensor. Later, we have obtained the form of Ricci curvature tensor, scalar curvature tensor and canhormanic curvature tensor on nearly cosypmlectic manifolds with generalized Tanaka-Webster connection. Finally, in the last section, we study conharmonically flat nearly cosypmlectic manifolds with generalized Tanaka-Webster connection.

PRELIMINARIES

Let (M,ϕ,η,ξ,g) be an $(n = 2m + 1)$-dimensional differentiable manifold M is called an almost contact Riemannian manifold, where ϕ is a (1; 1)-tensor field, ξ is the structure vector field, η is a 1-form and g is the Riemannian metric. This structure satisfies the following conditions,

$$\phi^2X = -X + \eta(X)\xi \tag{2.1}$$
$$\eta(\xi) = 1,\ \phi\xi = 0,\ \eta\cdot\phi = 0 \tag{2.2}$$
$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2.3}$$
$$g(X,\phi Y) = -g(\phi X,Y) \text{ and } g(X,\xi) = \eta(\xi) \tag{2.4}$$

for any vector fields X and Y on M [1].

With condition (2.5) an almost contact Riemannian manifold M is said to be a nearly cosymlectic manifold

$$(\nabla_X\phi)Y + (\nabla_Y\phi)X = 0 \text{ and } (\nabla_X\phi)X = 0 \tag{2.5}$$

It is said to be nearly cosymplectic manifold if the following conditions are satisfying,

$$\nabla_X\xi = HX, g(\nabla_X\xi,Y) + g(Y,\nabla_X\xi) = 0 \text{ and } (\nabla_X\phi)\xi = -\phi HX \tag{2.6}$$

where ∇ denotes the Levi-Civita connection.

In addition on a nearly cosymplectic manifold M, the following relations are hold [1].

$$(\nabla_X\eta)Y = g(\nabla_X\xi,Y) = g(HX,Y) \tag{2.7}$$
$$R(\xi,X,Y,Z) = -g((\nabla_XH)Y,Z) = \eta(Y)g(H^2X,Z) - \eta(Z)g(H^2X,Y) \tag{2.8}$$
$$\eta(R(Y,Z)X) = g((\nabla_XH)Y,Z) \tag{2.8}$$
$$S(X,\xi) = -\eta(X)tr(H^2) \tag{2.9}$$

where R is stated the Riemannian curvature tensor, S is shown the Ricci tensor and H is a skew-symmetric tensor field.

GENARALIZED TANAKA-WEBSTER CONNECTION ON NEARLY COSYMPLECTIC MANIFOLDS

The generalized Tanaka-Webster connection $\tilde{\nabla}$ on a almost contact metric manifold M is defined by

$$\tilde{\nabla}_XY = \nabla_XY - \eta(Y)\nabla_X\xi + (\nabla_X\eta)(Y)\xi - \eta(X)\phi Y \tag{3.1}$$

for all vector fields X and Y, where ∇ is Levi-Civita connection on M.

If we use (2.6) and (2.7) in (3.1) we get,

$$\tilde{\nabla}_XY = \nabla_XY - \eta(Y)HX + g(HX,Y)\xi - \eta(X)\phi Y \tag{3.2}$$

for all vector fields X and Y.

By taking $Y = \xi$ in (3.2) and using (2.6) we obtain,

$$\tilde{\nabla}_X\xi = 0 \tag{3.3}$$

Thus we can state that, the characteristic vector field of a nearly cosymplectic manifold is paralel vie generalized Tanaka-Webster connection. Let M be an n-dimensional nearly cosymplectic manifold. The curvature tensor $\tilde{R}$ of M with respect to the connection $\tilde{\nabla}$ is given by,

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X\tilde{\nabla}_YZ - \tilde{\nabla}_Y\tilde{\nabla}_XZ - \tilde{\nabla}_{[X,Y]}Z \tag{3.4}$$

Than, after a long computation in a nearly cosymplectic manifold, we have,

$$\begin{aligned}\tilde{R}(X,Y)Z = {} & R(X,Y)Z + \eta(X)(\nabla_Y\phi)Z - \eta(Y)(\nabla_X\phi)Z \\ & - \eta(X)\eta(Z)H^2Y + \eta(Y)\eta(Z)H^2X \\ & + \eta(X)\eta(Z)\phi HY - \eta(Y)\eta(Z)\phi HX \\ & + g(Z,HY)HX - g(Z,HX)HY \\ & - 2g(Y,HX)\phi Z + \eta(X)g(H^2Y,Z)\xi \\ & - \eta(Y)g(H^2X,Z)\xi + \eta(X)g(HY,\phi Z)\xi \\ & - \eta(Y)g(HX,\phi Z)\xi\end{aligned} \tag{3.5}$$

By taking $Z = \xi$ in (3.5) we get,

$$\tilde{R}(X,Y)\xi = R(X,Y)\xi - \eta(X)H^2Y + \eta(Y)H^2X \tag{3.6}$$

The Ricci tensor $\tilde{S}$ and the scalar curvature $\tilde{r}$ of a nearly cosymplectic manifold M with respect to the connection $\tilde{\nabla}$ is given by,

$$\begin{aligned}\tilde{S}(Y,Z) = {} & S(Y,Z) - \eta(Y)div(\phi)Z + \eta(Y)\eta(Z)tr(H^2) \\ & - \eta(Y)\eta(Z)tr(\phi H) + g(Z,HY)tr(H) \\ & + 2g(HY,\phi Z),\end{aligned} \tag{3.7}$$

$$\tilde{r} = r + (2m+1)tr(H^2),$$
$$\tilde{r} = 0. \tag{3.8}$$

By taking $Z = \xi$ in (3.7) we get,

$$\tilde{S}(Y,\xi) = S(Y,\xi) + \eta(Y)tr(H^2),$$
$$\tilde{S}(Y,\xi) = 0 \tag{3.9}$$

This gives

$$\tilde{Q}Y = QY + tr(H^2)Y, \tag{3.10}$$
$$\tilde{Q}Y = 0.$$

Theorem 1. For a nearly cosymplectic manifold M with generalized Tanaka-Webster connection $\tilde{\nabla}$,

a. $\tilde{R}(X,Y)Z = -\tilde{R}(Y,X)Z$,
b. $\tilde{R}(X,Y,Z,W) = -\tilde{R}(X,Y,W,Z)$,
c. $\tilde{R}(X,Y,Z,W) - \tilde{R}(Z,W,X,Y) = \eta(X)g((\nabla_Y\phi)Z,W) - \eta(Y)g((\nabla_X\phi)Z,W) - \eta(Z)g((\nabla_W\phi)X,Y) + \eta(W)g((\nabla_Z\phi)X,Y) + 2\eta(X)\eta(Z)g(\phi HY,W) - 2\eta(Y)\eta(Z)g(\phi HX,W) - 2\eta(Y)\eta(W)g(\phi HX,Z) - 2\eta(X)\eta(W)g(\phi HY,Z) + 2g(W,HZ)g(\phi X,Y) - 2g(Y,HX)g(\phi Z,W)$,
d. $\tilde{R}(X,Y)Z + \tilde{R}(Y,Z)X + \tilde{R}(X,Z)Y = 2\eta(Y)(\nabla_Z\phi)X + 2\eta(Y)\eta(Z)H^2X - 2\eta(X)\eta(Y)H^2Z - 2\eta(Y)\eta(Z)\phi HX + 2\eta(X)\eta(Y)\phi HZ - 2g(Z,HX)HY + 2g(Y,HZ)\phi X + 2g(X,HZ)\phi Y + 2g(X,HY)\phi Z + 2\eta(X)g(H^2Z,Y)\xi - 2\eta(Z)g(H^2X,Y)\xi + 2\eta(Y)g(HZ,\varphi X)\xi$,
f. The Ricci tensor $\tilde{S}$ is not symmetric.

In a nearly cosymplectic manifold M of dimension $n > 2$, the conharmonic curvature tensor $\tilde{K}$ with respect to the generalized Tanaka-Webster connection $\tilde{\nabla}$ is given by

$$\tilde{K}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{(n-2)}\Big[\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y + g(Y,Z)\tilde{Q}X - g(X,Z)\tilde{Q}Y\Big] \tag{3.11}$$

for all vector fields X, Y and Z on M, where $\tilde{R}$, $\tilde{S}$ and $\tilde{Q}$ are the Riemannian curvature tensor, Ricci tensor and Ricci operator, respectively with respect to the connection $\tilde{\nabla}$.

Using (3.5), (3.7) and (3.9) in (3.11), we get

$$\begin{aligned}\tilde{K}(X,Y)Z =& R(X,Y)Z + \eta(X)(\nabla_Y\phi)Z - \eta(Y)(\nabla_X\phi)Z \\ & -\eta(X)\eta(Z)H^2Y + \eta(Y)\eta(Z)H^2X \\ & +\eta(X)\eta(Z)\phi HY - \eta(Y)\eta(Z)\phi HX \\ & +g(Z,HY)HX - g(Z,HX)HY \\ & -2g(Y,HX)\phi Z + \eta(X)g(H^2Y,Z)\xi \\ & -\eta(Y)g(H^2X,Z)\xi + \eta(X)g(HY,\phi Z)\xi \\ & -\eta(Y)g(HX,\phi Z)\xi \\ & -\frac{1}{(n-2)}[S(Y,Z)X - S(X,Z)Y + \eta(X)div(\phi)ZY \\ & -\eta(Y)div(\phi)ZX - \eta(X)\eta(Z)tr(H^2)Y \\ & +\eta(Y)\eta(Z)tr(H^2)X + \eta(X)\eta(Z)tr(\phi H)Y \\ & -\eta(Y)\eta(Z)tr(\phi H)X + g(Z,HY)tr(H)X \\ & -g(Z,HX)tr(H)Y - g(X,Z)tr(H^2)Y \\ & +g(Y,Z)tr(H^2)X + 2g(HY,\phi Z)X \\ & -2g(HX,\phi Z)Y + g(Y,Z)QX \\ & -g(X,Z)QY],\end{aligned} \tag{3.12}$$

And thus,

$$\begin{aligned}\tilde{K}(X,Y)Z =& K(X,Y)Z + \eta(X)(\nabla_Y\phi)Z \\ & -\eta(Y)(\nabla_X\phi)Z - \eta(X)\eta(Z)H^2Y \\ & +\eta(Y)\eta(Z)H^2X + \eta(X)\eta(Z)\phi HY \\ & -\eta(Y)\eta(Z)\phi HX + g(Z,HY)HX \\ & -g(Z,HX)HY - 2g(Y,HX)\phi Z \\ & +\eta(X)g(H^2Y,Z)\xi - \eta(Y)g(H^2X,Z)\xi \\ & +\eta(X)g(HY,\phi Z)\xi - \eta(Y)g(HX,\phi Z)\xi \\ & -\frac{1}{(n-2)}[\eta(X)div(\phi)ZY - \eta(Y)div(\phi)ZX \\ & -\eta(X)\eta(Z)tr(H^2)Y + \eta(Y)\eta(Z)tr(H^2)X \\ & +\eta(X)\eta(Z)tr(\phi H)Y - \eta(Y)\eta(Z)tr(\phi H)X \\ & +g(Z,HY)tr(H)X - g(Z,HX)tr(H)Y \\ & -g(X,Z)tr(H^2)Y + g(Y,Z)tr(H^2)X \\ & +2g(HY,\phi Z)X - 2g(HX,\phi Z)Y],\end{aligned} \tag{3.13}$$

In a nearly cosymplectic manifold, using (3.5), (3.7) and (3.9) in (3.11), we get

$$\begin{aligned}K(\xi,Y)Z =& (\nabla_Y\phi)Z - \eta(Y)(\nabla_\xi\phi)Z + \eta(Z)\phi HY + g(HY,\phi Z)\xi \\ & -\frac{1}{(n-2)}[S(Y,Z)\xi + div(\phi)ZY - \eta(Y)div(\phi)Z\xi \\ & +\eta(Y)\eta(Z)tr(H^2)\xi + \eta(Z)tr(\phi H)Y \\ & -\eta(Y)\eta(Z)tr(\phi H)\xi + g(Z,HY)tr(H)\xi \\ & +2g(HY,\phi Z)\xi],\end{aligned} \tag{3.14}$$

And

$$\begin{aligned}\eta(K(X,Y)Z) =& -\frac{1}{(n-2)}[S(Y,Z)\eta(X) - S(X,Z)\eta(Y) \\ & +\eta(X)div(\phi)\eta(ZY) - \eta(Y)div(\phi)\eta(ZX) \\ & +g(Z,HY)tr(H)\eta(X) - g(Z,HX)tr(H)\eta(Y) \\ & +2g(HY,\phi Z)\eta(X) - 2g(HX,\phi Z)\eta(Y)].\end{aligned} \tag{3.15}$$

CONHARMONICALLY FLAT NEARLY COSYMPLECTIC MANIFOLDS WITH RESPECT TO THE CONNECTION $\tilde{\nabla}$

Assume that, M is conharmonically flat nearly cosymplectic manifold with respect to the connection $\tilde{\nabla}$. That is, $\tilde{K} = 0$. Then from (3.11), we have

$$\tilde{R}(X,Y)Z = \frac{1}{(n-2)}[\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y + g(Y,Z)\tilde{Q}X - g(X,Z)\tilde{Q}Y]. \tag{4.1}$$

Taking inner product with ξ in (4.1), then

$$g(\tilde{R}(X,Y)Z,\xi) = \frac{1}{(n-2)}[\tilde{S}(Y,Z)\eta(X) - \tilde{S}(X,Z)n(Y) + g(Y,Z)\tilde{S}(X,\xi) - g(X,Z)\tilde{S}(Y,\xi)] \tag{4.2}$$

This gives,

$$-\tilde{R}(X,Y,\xi,Z) = \frac{1}{(n-2)}[\tilde{S}(Y,Z)\eta(X) - \tilde{S}(X,Z)n(Y) + g(Y,Z)\tilde{S}(X,\xi) - g(X,Z)\tilde{S}(Y,\xi) \tag{4.3}$$

Using (3.7) in (4.3), we get

$$\begin{aligned} -\tilde{R}(X,Y,\xi,Z) = \frac{1}{(n-2)}[&S(Y,Z)\eta(X) - S(X,Z)n(Y) \\ &+n(X)g(Z,HY)tr(H) \\ &-n(Y)g(Z,HX)tr(H) \\ &+2\eta(X)g(HY,\phi Z) \\ &-2\eta(Y)g(HX,\phi Z)]. \end{aligned} \tag{4.4}$$

Using (2.8), (3.6) in (4.4) and taking X = ξ we have

$$S(Y,Z) = -g(Z,HY)tr(H) - 2g(HY,\phi Z) - \eta(Y)\eta(Z)tr(H^2), \tag{4.5}$$

And

$$r = -(2m+1)tr(H^2) \tag{4.6}$$

Thus, we can state the following:

Theorem 2. For a conharmonically flat nearly cosymplectic manifold with respect to generalized Tanaka-Webster connection, the scalar curvature is $(2m+1)tr(H^2)$.

CONCLUSION

With this study, we have demonstrated the important curvature properties of nearly cosymplectic manifolds equipped with Tanaka-Webster connection. Also, based on these curvature properties, we have defined the conhormanic curvature tensor. Then, we have emphasized the properties that canhormanic curvature tensor provides in case of flatness. In addition to this work, conharmonically symmetric, ϕ-conharmonicallyflat, ξ-conharmonicallyflat nearly cosymplectic manifolds equipped with Tanaka-Webster connection can be examined.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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