



Research Article

Flapwise vibration of functionally graded rotating tapered beam with tipmass

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ABSTRACT

A flapwise vibration analysis is performed on a functionally graded tapered rotating beam with tip mass. The beam is considered as functionally graded, whose material properties varying in the direction of thickness. It is assumed as beam is a taper beam with uniform width. Using Mori Tanaka method the properties of the functionally graded material are achieved. The first three frequencies of flapwise are obtained using finite element analysis based software ANSYS. By comparing with the results in open literature, the accuracy of the results is tested. The proposed method yielded good results compared with the existing Frobenius method because of good mesh capabilities. Different parametric studies are carried out to investigate the influence of tip mass on the functionally graded (FG) rotating taper beam flapwise frequencies. The influence of rotating speed, hub radius ratio, gradient index, taper ratio and hub radius ratio on flapwise frequency are elaborated with the support of ANSYS results and analytical models. The results show that the variation in tip mass tends to depress the frequencies from 107.19 Hz to 25.31 Hz at 100 rps in fundamental frequency and in third frequency it is decreased from 336.70 to 332.13 Hz at the same rotational speed.

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INTRODUCTION

Rotating beam structures are commonly used in various engineering fields such as rotating machines and turbo machinery. The components like helicopter blades, wind mill blades, aero plane propellers can be considered

as rotating cantilever beams. Tapered blades are preferred in rotating machines to save material weight and decrease stress. Due to their unique characteristics, such as high stiffness and heat resistance, functionally graded materials

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are used for the rotating blades. Due to action of centrifugal force, predicting the vibration characteristics of rotating structures is of interest to many researchers. Several studies are done on rotating taper beam with homogeneous materials. Banerjee and Sobey [1] developed energy expressions for the Timoshenko rotating taper beam using the differential transformation method, Ozdemir and Kaya [2, 3] investigated the effect of different parameters on the flapwise frequency of the rotating taper beam. The free vibration of rotating tapered beams was studied by Benerjee and Jackson [4] using the dynamic stiffness method. An explicit expression was presented by Bazone [5] to evaluate the effect of tapering on the rotating beam natural frequency. Shahba et. al. [6] carried out a study of free vibration and stability of axially functionally graded taper beams using finite element method (FEM) approach. Zarrinzadeh's et. al. [7] proposed a new finite element model to analyze vibration of axially functionally graded rotating tapered beams for different boundary conditions. Auciello [8] demonstrated boundary characteristic technique orthogonal polynomial technique to study the vibration behavior of rotating beam with non-uniform tapers. The effect of the taper ratio on the parametric stability of the rotating taper beam using the monodromy matrix method was investigated by Bulut [9]. Rajasekaran [10] applied differential transformation and quadrature methods to analyze free vibration of the axially functionally graded Timoshenko beam. Flapwise bending vibration of doubly tapered dynamically graded beam using Raleigh Ritz method based on the hybrid deformation variables was proposed by Ramesh and Mohan Rao [11]. Adair and Jaeger [12] demonstrated Adomian decomposition method to perform vibration analysis of non-uniform rotating Euler-Bernoulli beam subjected to different boundary conditions.

In order to increase the efficiency, tip masses are introduced to rotating structures. Tip mass helps to adjust the vibration frequency, to increase the air flow and flexing motion of the wind turbine blades and helicopter rotors. It is also used to provide auto-cooling in the case of turbine blades, airplane wings, and missile fans. Therefore, several researchers have studied the dynamic behavior of the rotating beam with tip mass. By using a singular perturbation technique, William and Handelman [13] obtained an approximate solution for a rotating beam with tip mass. To estimate the transverse vibration of a rotating beam with tip mass, Jones [14] applied an integral equation method. Hoa [15] investigated the vibration behavior of a rotating beam with tip mass using the finite element method. Wright et. Al. [16] using the Frobenius method, calculated the natural frequencies of a rotating beam with tip mass. Bhat [17] examined transverse vibration of rotating tip mass cantilever beam using Rayleigh-Ritz method based on characteristic orthogonal polynomials. Yoo et. al. [18, 19] presented vibration analysis using the hybrid deformation-based simulation technique of rotating pre-twisted blades

with concentrated mass. Lin et. al. [20] derived differential equations for pre twisted non uniform rotating beam with tip mass and proposed closed form solution using mode expansion method. Lee et. al. [21] investigated effect of tip mass and other parameters on frequency of pre twisted non uniform rotating beam using semi analytical transition matrix based algorithm.

Liao and Wang [22] applied Mori-Tanaka homogenization technique to determine material properties of Functionally Graded Material (FGM) micro beams are assumed to vary in the thickness direction. The free vibration response of a rotating tapered composite beam with tip mass was studied by Vasudevan and Pardhasaradhi [23]. For functionally graded beams with concentrated mass, Ramesh and Mohan Rao [24] performed chordwise vibration analysis. Ravi Kumar et. al. [25, 26] investigated the flapwise vibration of functionally graded rotating beams using differential transform method. Aksencer and Aydogdu [27] examined free vibration of composite with point mass using algebraic polynomials based Ritz method. Chen and Jingtao [28] presented a fourier series solution to analyse vibration of a rotating beam with elastic boundary conditions. Malik and Das [29] performed vibration analysis of rotating nano beams using Ritz method by considering spin softening and Coriolis Effect. Li and Cheng [30] investigated the frequencies and mode shapes of a rotating twisted blade using orthogonal polynomials developed based on shallow shell theory. Reviewing the literature, most previous studies have studied the effect of tip mass on isotropic beams and relatively very few young researchers have examined the structural behavior of FG beams whose material properties change in the direction of thickness.

The aim of the current work is to examine the flapwise vibration of the tip mass of the functionally graded rotating taper beam. In the analysis FG beam properties are obtained using the Mori Tanaka method and fundamental frequencies are obtained using ANSYS20 software developed on the principles of the finite element method. The natural frequencies of stretching motion are far greater than the bending motion and the coupling effect become negligible for the slender beams. Hence, the stretching equation of motion and coupling effect between the stretching and bending motions are ignored [31]. Therefore, the coupling effects were neglected in this work. The effect of tip mass on natural frequency, with respect to rotating speed, taper ratio, gradient index and hub radius ratio is discussed. To bring out the efficiency of the proposed method, results of current investigation are compared with existing results reported in the previous study.

PROBLEM MODELLING

The current study contributes to the free vibration characteristics and analysis of functionally graded tapered

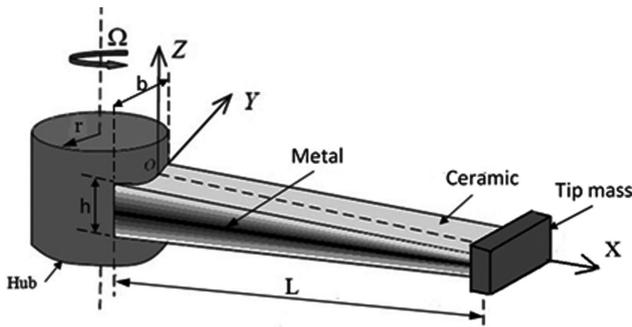


Figure 1. Functionally graded tapered rotating beam with tip Mass.

rotating beam with tip mass. As shown in Figure 1, the beam consists of a metallic core - ceramic surfaces. The material properties changes symmetrically from the core to the surface in the direction of thickness. The beam is made functionally graded material with rich ceramic content on the upper (Top) and lower surfaces (Bottom) i.e. at $z = -h/2$ and $h/2$ and with a protective metallic core at middle surface (at $z = 0$).

The following assumption are made for the taper beam

$$A(x) = A_h \left(1 - \frac{cx}{L}\right)^m \quad (1)$$

$$I_y(x) = I_{yh} \left(1 - \frac{cx}{L}\right)^{m+2} \quad (2)$$

$$I_z(x) = I_{zh} \left(1 - \frac{cx}{L}\right)^{m+2} \quad (3)$$

In the taper assumptions 'A' represents the cross-sectional area, I_y , I_z denotes moment of inertia about the axes Y and Z respectively. The 'h' subscript represents the values indicated at hub position (Figure 1). 'c' is the taper ratio which is less than 1. Otherwise the beam cross section tapers to zero between ends. 'm' indicates the taper variation in the cross section. In most practical case linear variation of the area and cubic variation of the moment of inertia are exist. So, $m = 1$ is taken in taper assumption.

GEOMETRIC MODELING

The beam dimensions of a length of $L = 1000$ mm, a width of $b = 20$ mm and a thickness of $h = 10$ mm are considered for the present study. Since the material is ceramic - metallic core FGM, the material properties of the beam vary symmetrically from core to surface in the direction of thickness. Using the Mori Tanaka method, the effective material properties are evaluated. The equivalent bulk

Table 1. Metal and Ceramic constituent material properties

Name of the Property	Metal (Steel)	Ceramic (Alumina)
Density (kg/m^3)	7800	3200
Young's Modulus (GPa)	214	390
Modulus of Rigidity (GPa)	82.2	137
Poisson's ratio	0.3	0.34

modulus (K_{eq}) and shear Modulus (G_{eq}) of the FG beam as per the Mori Tanaka method is given by [22]

$$\frac{K_{eq} - K_{mt}}{K_{cr} - K_{mt}} = \frac{V_{cr}}{1 + V_{mt}(K_{cr} - K_{mt})/K_{mt} - 4G_{mt}/3} \quad (4)$$

$$\frac{G_{eq} - G_{mt}}{G_{cr} - G_{mt}} = \frac{V_{cr}}{1 + V_{mt}(G_{cr} - G_{mt})/[G_{mt} + G_{mt}(9G_{mt} - 8G_{mt})/6(K_{mt} + 2G_{mt})]} \quad (5)$$

$$V_{cr}(z) = \left(\frac{2z}{h}\right)^n, V_{mt} = 1 - V_{cr} \quad (6)$$

Where 'n' is known as gradient index. The subscripts 'cr' and 'mt' represents ceramic and metal properties respectively. Now the expressions for the effective young's modulus 'E' and poisons ratio ' μ ' are as follows

$$E(z) = \frac{9K_{eq}G_{eq}}{3K_{eq} + G_{eq}} \quad (7)$$

$$\mu(z) = \frac{3K_{eq} - 2G_{eq}}{6K_{eq} + 2G_{eq}} \quad (8)$$

According to rule of mixture the expression for effective mass density ' ρ ' is

$$\rho(z) = \rho_{cr}V_{cr} + \rho_{mt}V_{mt} \quad (9)$$

The mechanical properties of the constituent materials are represented in Table 1.

FINITE ELEMENT MODELING

The problem is modeled in ANSYS20 Software and the finite element mesh is generated for Solid 186 element which represents the beam model. The mesh refinement is carried out until the solution to be converged. In the convergence study the first three flapwise frequencies are

Table 2. Comparison of frequency ratios of Present Approach with Frobenius method for the non-rotating uniform beam

Mass ratio	First Frequency		Second Frequency	
	Present	Frobenius [16]	Present	Frobenius [16]
1	1.5635	1.5573	16.3747	16.2500
0.5	2.0169	2.0163	16.9022	16.9014
2	1.1585	1.1582	15.8622	15.8609

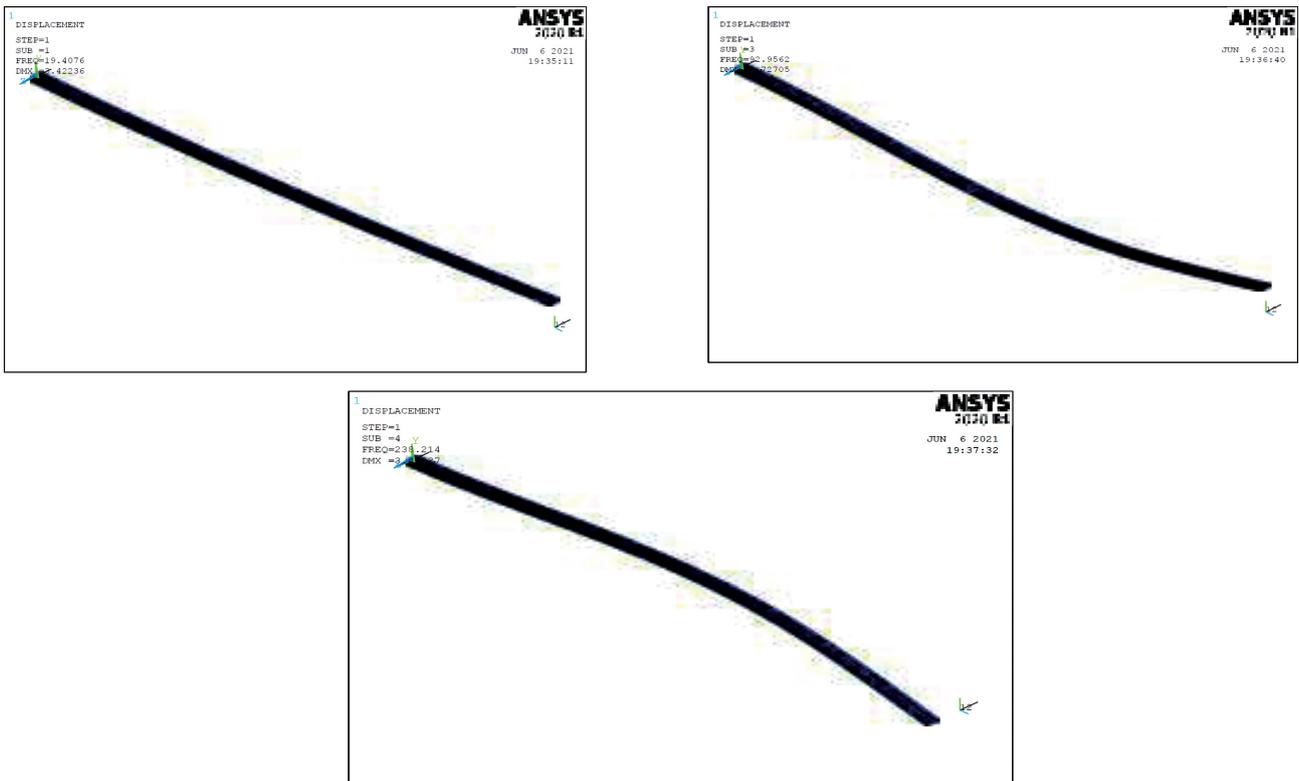


Figure 2. Mode shapes of functionally graded rotating taper beam.

Table 3. Flapwise frequency of functionally graded rotating beam with various tip mass ratios and rotating speeds at $c = 0.5, \delta = 0, n = 0$

Ω (in rps)	First Frequency in Hz			Second Frequency in Hz			Third Frequency in Hz		
	mr = 0	0.5	2	mr = 0	0.5	2	mr = 0	0.5	2
0	19.408	8.7315	4.7216	92.956	66.078	63.418	238.21	193.94	191.33
20	29.583	13.347	7.1943	104.63	73.522	70.708	242.92	204.02	201.45
40	47.899	21.644	11.6	133.64	91.833	88.596	256.54	231.33	228.8
60	67.461	30.392	16.209	171.14	115.16	111.34	277.76	269.98	267.45
80	87.285	39.12	20.787	212.41	140.44	135.98	304.98	304.85	302.94
100	107.19	47.77	25.312	255.47	166.47	161.36	336.70	336.42	332.13

determined with the present model with respect to different numbers of elements. It is observed that the present model converges rapidly at number of elements 15 in length direction. Hence for further analysis the beam is divided into 15 equal length elements based on the procedure adopted by Zarrinzadeh, H et al. [32]. A solid element is created at the beam tip which represents the tip mass. One end of the beam is constrained to all degrees of freedom which represents the rotating hub end. Inertial load of angular velocity is applied for rotating beam according to rotating speed. The density of tip mass is modified according to mass ratio.

DISCUSSION ON NUMERICAL RESULTS

The confirmity of the present methodology is verified by relating the results with the available literature. By assuming the metallic beam ($n \rightarrow \infty$) the first two frequencies are obtained. The beam dimensions of a length of $L = 1000$ mm, a width of $b = 20$ mm and a thickness of $h = 10$ mm at the fixed end. The cross section is varies according to taper ratio. For comparison of results with uniform beam the taper ratio is set to zero in the equations

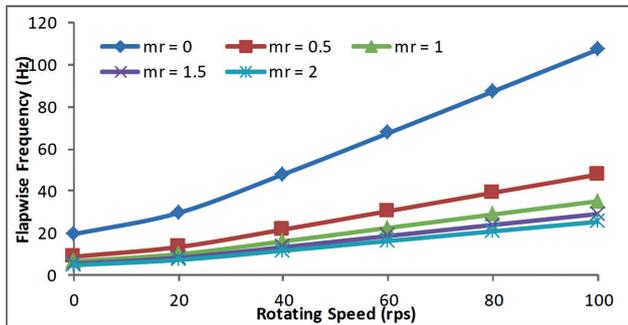


Figure 3. First flap wise frequency variation with Rotating speed for different mass ratios (mr).

1-3. The results are compared in Table 2 for various mass ratios (mr). Table 2 represents a good agreement between the FEM based present approach and Frobenius method. Hence it was assumed that present approach is appropriate for further study.

Further analysis is carried up with same dimensions of the beam which is used for the validation purpose. Beam is considered as functionally graded with steel and alumina is constituent materials. The mechanical properties of steel (metallic constituent) and alumina (ceramic constituent) are given in Table 1. The equivalent mechanical properties are obtained from equation 4-8. The functionally graded beam is modeled in ANSYS and the mode shapes are represented in Figure 2. For different mass ratios and rotating speed, the first three flapwise frequencies of functionally graded Rotating Beam with tip mass are obtained and represented in Table 3. The taper ratio is set to 0.5, hub radius ratio is considered as 0. Gradient index is assumed as 0. The first flapwise frequency variation with rotating speed for different mass ratios is shown in Figure 3. It is found that with an increase in rotating speed, the flapwise frequency increases. This is due to stiffening effect caused by increase in centrifugal tension force.

Figure 3 also shows the effect of tip mass on the natural frequencies of the functionally graded rotating beam. The flapwise frequency is decrease with increase in mass ratio. The mass of the structure is increased with addition of tip mass. So stiffness of structure decreases causes decrease in frequency. On the other hand, due to the rotational speed of the beam at higher speeds, the stiffness of the structure is found to be enhanced with increase in tip mass and dominates effect of mass increase in the structure. Hence the rate of decrease is decreases with increase of speed.

The first three flap wise frequencies at different gradient index and mass ratios at constant rotational velocity of 50 rps and with hub radius ratio of 0 are obtained and illustrated in Table 4. The taper ratio is set to 0.5. The results

Table 4. Flapwise frequency of functionally graded rotating beam with various tip mass ratios and gradient index at $c = 0.5, \delta = 0, \Omega = 50$ rps

N	First Frequency in Hz			Second Frequency in Hz			Third Frequency in Hz		
	mr = 0	0.5	2	mr = 0	0.5	2	mr = 0	0.5	2
0	57.623	26.01	13.904	151.72	103.12	99.606	305.49	266.27	265.95
2	54.193	24.239	12.869	130.8	86.05	83.353	233.81	191.2	189.67
4	53.843	24.037	12.746	128.98	84.397	81.782	227.29	185.73	184.29
6	53.708	23.959	12.697	128.29	83.763	81.179	224.83	183.65	182.25
8	53.636	23.909	12.67	127.94	83.424	80.86	223.54	182.56	181.18
10	53.591	23.885	12.654	127.71	83.217	80.662	222.75	181.89	180.52

represent effect of the gradient index on the variation of flap wise frequencies of functionally graded taper beam with tip mass.

In Figure 4 the influence of gradient index on first flap wise frequency of functionally graded rotating taper beam with tip mass is observed. With an increase in the gradient index, the frequency decreases. The increase of non-homogeneity leads to soften the beam. So frequency drop occurs. The non-homogeneity effects up to 4 only. Addition of tip mass leads to decrease the efficiency. The rate of decrease in efficiency decreases with respect increase in mass due to increment of stiffness.

The effect of taper ratio on flapwise frequency of functionally graded beam with tip mass is investigated by performing the simulation at various taper ratio and different mass ratio. The first three flapwise frequencies are at constant speed 50 rps with zero gradient index and hub radius ratio are obtained and represented in Table 5.

The first flapwise frequency variation with taper ratio at different mass ratios is plotted in Figure 5. As the taper ratio increases, it is found that frequencies decrease since

the rising taper ratio has a softening effect resulting from the reduction of the cross-sectional area. The addition of tip mass also causes the decrease in stiffness. In the case of zero tip mass reverse trend is observed. It represents the taper ratio effect for fundamental frequency is minimum.

The first three flap wise frequencies at different hub ratio and mass ratios are obtained and represented in Table 6. The taper ratio is set to 0.5. The gradient index is assumed as zero and rotating speed is 50 rps. The Results represents the effect of hub radius ratio on frequency of functionally graded taper beam with tip mass.

The first flapwise frequency variation of functionally graded tapered rotating beam with hub radius ratio is represented for different mass ratios in Figure 6. The increase in hub radius ratio increases the frequency. This is because of the beam's root is position. It is farther away from the center of rotation and it increases the centrifugal force beam, which increases the beam's stiffness. On the other hand, the addition of tip mass reduces the beam stiffness, which leads the frequency to decrease.

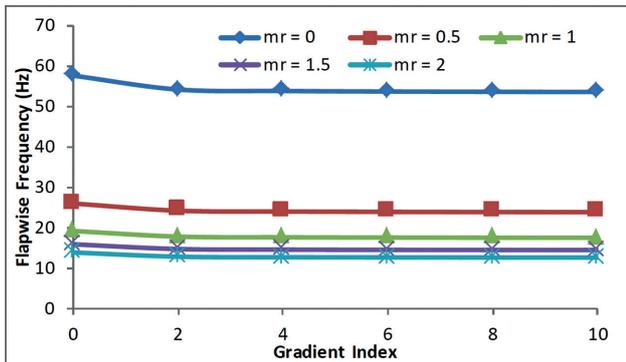


Figure 4. First flap wise frequency variation with gradient index at different mass ratios (mr) at $c = 0.5$, $\delta = 0$, $\Omega = 50$ rps.

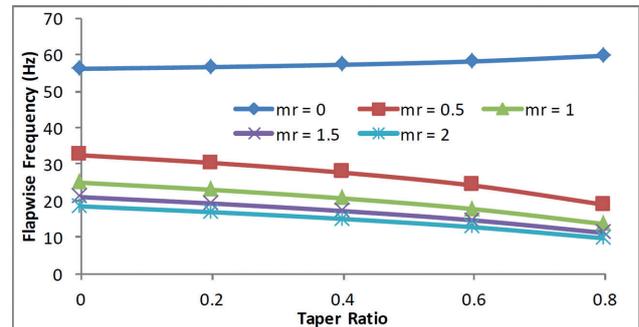


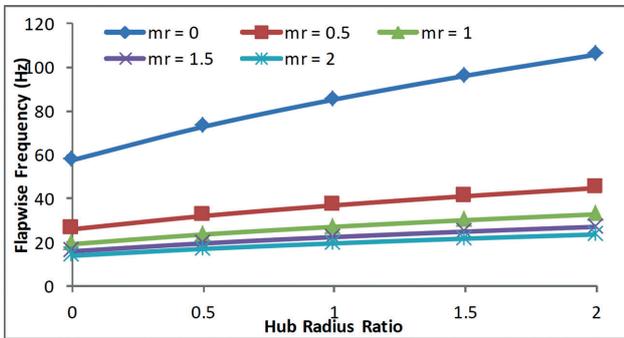
Figure 5. First flap wise frequency variation with gradient index at different mass ratios (mr) at $c = 0.5$, $\delta = 0$, $\Omega = 50$ rps.

Table 5. Flapwise frequency of functionally graded rotating beam with various taper ratio, tip mass ratios at $n = 0$, $\delta = 0$, $\Omega = 50$ rps

C	First Frequency in Hz			Second Frequency in Hz			Third Frequency in Hz		
	mr = 0	0.5	2	mr = 0	0.5	2	mr = 0	0.5	2
0	56.107	32.413	18.434	169.24	125.04	117.67	376.81	318.11	311.75
0.2	56.576	30.31	16.837	162.61	116.84	110.98	349.51	291.9	287.17
0.4	57.21	27.644	14.974	155.5	107.94	103.63	320.67	264.22	263.18
0.6	58.134	24.092	12.705	147.76	97.958	95.236	289.65	269.81	269.56
0.8	59.673	18.851	9.6688	139.38	85.952	84.753	282.51	282.51	282.4

Table 6. Flapwise frequency of functionally graded rotating beam with various tip mass ratios and hub radius ratio at $c = 0.5$, $n = 0$, $\Omega = 50$ rps

Δ	First Frequency in Hz			Second Frequency in Hz			Third Frequency in Hz		
	mr = 0	0.5	2	mr = 0	0.5	2	mr = 0	0.5	2
0	57.623	26.01	13.904	151.72	103.12	99.606	305.49	266.27	265.95
0.5	72.835	32.053	17.033	179.82	122.26	118.52	342.52	283.82	283.51
1	85.307	36.922	19.549	203.87	138.3	134.34	375.51	309.27	306.72
1.5	96.132	41.103	21.709	225.24	152.35	148.18	405.51	334.42	331.82
2	105.83	44.82	23.626	244.66	164.98	160.62	433.21	357.46	354.81

**Figure 6.** First flap wise frequency variation with hub radius ratio at different mass ratios (mr) at $n = 0$, $\Omega = 50$ rps.

CONCLUSIONS

The flapwise vibration of a functionally graded rotating cantilever taper beam with tip mass was studied in the present work. Using the Mori Tanaka method, the behavioral characteristics of the functionally graded beam material are obtained. Flapwise frequencies are obtained using finite element analysis (FEA) software ANSYS. The results obtained are validated by comparing with the results in open literature to verify the accuracy of the present method. The contribution of taper ratio, rotating speed, tip mass ratio and hub radius ratio on flapwise frequencies of functionally graded rotating tapered beam was studied. The results show that the variation in tip mass tends to depress the frequencies from 107.19 Hz to 25.31 Hz at 100 rps in fundamental frequency and in third frequency it is decreased from 336.70 to 332.13 Hz at the same rotational speed. These results indicate that natural frequencies of flap wise bending increase with an increase in rotating speed due to an increase in stiffness. The flapwise bending frequencies of the functionally graded rotating taper beam decreases due to softening of the beam as material non-homogeneity increases. If the taper ratio increases, the frequencies decrease since the increase in taper ratio has a softening

effect resulting from the reduction of the cross-sectional area. Since the root of the beam is positioned farther from the rotating centre, it has an effect on the centrifugal force. The increase in centrifugal force on the beam will cause a raise in the stiffness of the structure. Hence, the frequency increases with the hub radius. The tip mass increases the structure's mass, resulting in a reduction in frequency at constant speeds. The addition of tip mass increases the structure's mass which leads to drop in frequency. But, at higher speeds the rotating speed effect dominating the effect due to increase in the mass of the structure. Hence the rate of decrease in frequency is decreases with increase of speed.

NOMENCLATURE

A	cross sectional area of the beam
b	width of the beam
c	taper ratio
E	Young's modulus
G	shear modulus
h	thickness of the beam at hub
I	moment of inertia
K	bulk modulus
L	length of the beam
m	taper variation in the cross section
mr	mass ratio
n	gradient index
r	radius of hub
V	volume fraction
x	longitudinal coordinate
z	lateral coordinate
ρ	mass density
δ	hub radius ratio
Ω	rotating speed

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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